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For the degree of Doctor of Philosophy

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HOMOLOGY OPERATIONS IN THE SPECTRAL SEQUENCE OF A COSIMPLICIAL SPACE

A Dissertation

Submitted to the Faculty

of

Purdue University

by

Philip J. Hackney

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SYMBOLS

$\mathcal{C}$	A fixed E_∞ operad
$\Bbbk$	$\mathbb{Z}/2$, the field
π	$\mathbb{Z}/2 = \{e, \sigma\}$, the group
$\Bbbk^{\mathrm{triv}}$	$\Bbbk$ with trivial π-action
W	Minimal $\Bbbk[\pi]$-resolution of $\Bbbk^{\mathrm{triv}}$ (p. 3)
$\mathrm{sk}_t(C)$	Brutal truncation of the complex C above dimension t
$\Sigma(C^*)$	The cochain complex with $\Sigma(C)^{q+1} = C^q$
$\Sigma(C_*)$	The chain complex with $\Sigma(C)_{q+1} = C_q$
$\mathcal{E}(C)$	$W \otimes_\pi (C \otimes C)$
$\mathcal{E}^n(C)$	$\mathrm{sk}_n W \otimes_\pi (C \otimes C)$
Z_t	The group of cycles in C_t
B_t	The group of boundaries
$T(B)$	Product total complex of a bicomplex (p. 2)
$[p]$	The ordered set $\{0 < 1 < \cdots < p\}$
Δ^p	Normalized simp. chains for the standard p-simplex (p. 9)
Λ^p_r	The set of injections $[r] \hookrightarrow [p]$ which take 0 to 0 (p. 10)
$\zeta^p(S)$	The injection $[\lvert S\rvert - 1] \hookrightarrow [p]$ whose image is S
$C(Y)$	Conormalization:
	$C(Y)^p = X^p / d^1 Y^{p-1} + \cdots + d^p Y^{p-1}$
d	The homological differential in $C(Y)$
$đ$	The cosimplicial differential in $C(Y)$
D^p_{rst}	$\Sigma^{t-s}\,\mathrm{coker}(\mathrm{sk}_{s-1}\Delta^p \to \mathrm{sk}_{s+r-1}\Delta^p)$ (p. 12)
Θ_y	Map of cosimplicial chain complexes $D_{rst} \to Y$ (p. 15)
$\Omega_{s,s'}$	Cochain complex $C(H_s(D_{\infty ss}) \otimes H_{s'}(D_{\infty s' s'})))$
$\omega^p_{s,s'}$	A basis for $\Omega_{s,s'}$ (p. 30)

$\bar{\Omega}_s$	$\Omega_{s,s}/\pi$
$Z^r_{p,q}$	$\{\, x \in F^p C_{p+q} \mid \partial x \in F^{p-r} C_{p+q-1} \,\}$
$B^r_{p,q}$	$\partial Z^{r-1}_{p+r-1,q-r+2} + Z^{r-1}_{p-1,q+1}$
$E^r_{p,q}$	$Z^r_{p,q}/B^r_{p,q}$
δ^r	Spectral sequence differential on page r
$[v]_r$	The class that v represents on page r, where $v \in E^{r'}$, $r' \leq r$
$\mathfrak{q}_{p,q}$	Generator for bidegree (p, q) of some spectral sequence (p. 48)
$\hat{Q}^m$	External Operations defined on r-cycles (p. 49)
$\tilde{Q}^m$	External Operations defined on E^r (p. 60)
$\widetilde{\lambda}_n$	External Browder operation (p. 70)

ABSTRACT

Hackney, Philip J. Ph.D., Purdue University, August 2010. Homology Operations in the Spectral Sequence of a Cosimplicial Space. Major Professor: James E. McClure.

We construct operations in the homology spectral sequence of cosimplicial E-infinity and cosimplicial E-n spaces. This is accomplished by constructing external operations for certain universal examples which were introduced by Bousfield and Kan. By universality we then have external operations for any cosimplicial space and the E-n structure maps provide the internal operations. The main ingredient is a detailed computation of the spectral sequence associated to the homotopy orbit of the square of the Bousfield-Kan examples.

1. INTRODUCTION

Let $\mathcal{C}$ be a fixed E_∞ operad. The aim of this work is to prove the following Theorem and its E_n analogue (Theorem 7.4.5).

Theorem 1.0.1. *Suppose that X is a cosimplicial object in the category of $\mathcal{C}$-spaces. Then there are operations in the mod-2 homology spectral sequence associated to X:*

$$Q^m : E^r_{-s,t} \to E^r_{-s,m+t} \qquad m \geq t$$
$$Q^m : E^r_{-s,t} \to E^w_{m-s-t,2t} \qquad m \in [t-s,t]$$

where $w \in [r, 2r-2]$ is given in Theorem 6.3.6.

The first proof of this was given by Jim Turner in [1]. We provide a fundamentally different proof which is more direct and is amenable to generalization.

Pictorially, the images of these operations applied to an element in bidegree $(-s,t)$ lie on the dotted and solid lines of Figure 1.1.

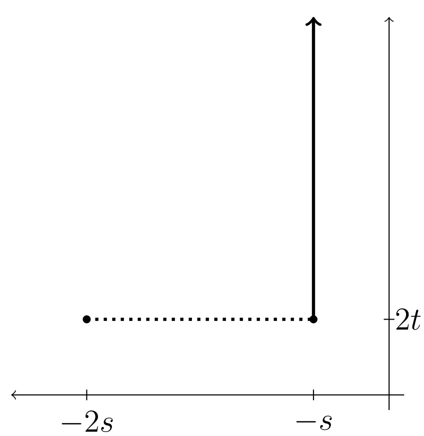

Figure 1.1. Vertical and Horizontal Operations

The scheme of the paper is as follows. We work with simple *Bousfield-Kan universal examples*, which one might think of as cosimplicial spheres. We construct *external operations* for these examples and use the universal property to transport these operations into the spectral sequence for an arbitrary cosimplicial space. When that cosimplicial space is actually a cosimplicial $\mathcal{C}$-space, we then obtain internal operations by combining the external operations with the $\mathcal{C}(2)$-structure.

1.1 Homology Spectral Sequence and Passage to Chains

Let X be a cosimplicial space (where 'space' means either topological space or simplicial set). We briefly outline the construction of the homology spectral sequence associated to X (see [2] for more details).

Convention. We always work over the field $\Bbbk = \mathbb{Z}/2$ and just write Ch for the category of chain complexes over $\Bbbk$.

Let

$$S_* : \text{Spaces} \to \text{Ch}$$

be the mod-2 chains functor. The first step in the construction of the spectral sequence is to pass from X to the cosimplicial chain complex $S_*(X)$.

If Y is a cosimplicial object in an abelian category $\mathcal{A}$, we will write CY for the conormalization of Y

$$CY^p = \operatorname{coker}\Big(\bigoplus_{k=1}^{p} d^k : \bigoplus Y^{p-1} \to Y^p\Big),$$

which is an object in $\text{coCh}^{\geq 0}\,\mathcal{A}$, the category of nonnegative cochain complexes over $\mathcal{A}$. When $\mathcal{A} = \text{Ch}$, the category of chain complexes over $\Bbbk$, we will regard CY as a left-plane bicomplex which consists of the $\Bbbk$-module CY^p_q in bidegree $(-p, q)$. Given a bicomplex B, we will let TB denote the product total complex:

$$TB_m = \prod_j B_{j,m-j}.$$

The appropriate filtration in this situation is the one by columns

$$F^k_m = \prod_{j \leq k} B_{j,m-j}.$$

We may regard

$$TC(Y) \subset \prod_m \mathrm{Hom}(\Delta_*^m, Y^m),$$

where this is the internal Hom in the category Ch. The natural filtration of Δ_* by skeleta induces the above filtration on $TC(Y)$ (see [2]).

The homology spectral sequence associated to X is, by definition, the one obtained for this filtration on $TCS_*(X)$. For this reason we usually work with cosimplicial chain complexes rather than cosimplicial spaces, though of course we will have to check that various geometric constructions we make behave well when we pass to chains. This will usually take the form of an E^1 or E^2 isomorphism between the algebraic and geometric constructions.

1.2 External Operations

In this section we recall that the Dyer-Lashof operations are constructed by combining an "external operation" with the $\mathcal{C}(2)$-structure map. We will also use this to give one construction of the vertical operations from Theorem 1.0.1.

Let W be the usual $\Bbbk\pi$-free resolution of $\Bbbk^{\mathrm{triv}}$, defined by

$$W_i = \begin{cases} \Bbbk\pi \cdot e_i & i \geq 0 \\ 0 & i < 0 \end{cases}$$

and

$$d(e_i) = (1+\sigma)e_{i-1}.$$

Of course this is $\Bbbk\pi$ chain-homotopic to $S_*(E\pi)$, which, combined with the shuffle map gives a quasi-isomorphism

$$W \otimes_\pi (S_*(X) \otimes S_*(X)) \to S_*(E\pi \times_\pi (X \times X))$$

for any space X. If X is a $\mathcal{C}$-space then $\mathcal{C}(2)$ is equivariantly homotopic to $E\pi$ and so there is a map

$$E\pi \times_\pi (X \times X) \to X$$

which induces

$$W \otimes_\pi (S_*(X) \otimes S_*(X)) \to S_*(X).$$

Let C be a chain complex. We define for each m a chain map of degree m

$$\begin{aligned} q^m : C &\to W \otimes_\pi (C \otimes C) \\ c &\mapsto e_{m-|c|} \otimes c \otimes c + e_{m+1-|c|} \otimes c \otimes dc \end{aligned}$$

(interpreting terms with e_{-n} as zero). If C is a chain complex equipped with a map $W \otimes_\pi (C \otimes C) \to C$ (for example if C is chains on a $\mathcal{C}$-space) then the image of $[c]$ under the composite

$$H_*(C) \to H_*(W \otimes_\pi (C \otimes C)) \to H_*(C)$$

is, by definition, $Q^m[c]$. For this reason we call the homology class

$$q^m[c] \in H_*(W \otimes_\pi (C \otimes C))$$

an 'external operation'.

Let's go through the same procedure in the cosimplicial case which will end up giving us the vertical operations. The tensor product of two cosimplicial chain complexes $A_*^\bullet$ and $B_*^\bullet$ is the cosimplicial chain complex given in cosimplicial degree p by $A_*^p \otimes B_*^p$. If X is a cosimplicial $\mathcal{C}$-space, then we have a map

$$W \otimes_\pi (S_*(X) \otimes S_*(X)) \xrightarrow{E^1 \text{ iso.}} S_*(E\pi \times_\pi (X \times X)) \to S_*X,$$

so it useful to consider cosimplicial chain complexes Y equipped with a map

$$W \otimes_\pi (Y \otimes Y) \to Y. \tag{1.1}$$

If Y is any cosimplicial chain complex then we have, for each m, a collection of maps

$$Y^p \to W \otimes_\pi (Y^p \otimes Y^p)$$

which constitute a (degree m) map of cosimplicial chain complexes

$$q^m : Y \to W \otimes_\pi (Y \otimes Y).$$

Combining this with (1.1) gives operations

$$Q^m : E^r(Y) \xrightarrow{q^m} E^r(W \otimes_\pi (Y \otimes Y)) \to E^r(Y).$$

Notice, though, that $[y] \in E^r_{-s,t}$ is mapped to something in bidegree $(-s, t+m)$ and to zero for $m < t$, so we only pick up the vertical part of Figure 1.1.

Henceforth, whenever we speak of external operations on a cosimplicial chain complex Y we will mean operations whose target is the spectral sequence for $W \otimes_\pi (Y \otimes Y)$. It will grow quite tedious to write

$$W \otimes_\pi (Y \otimes Y)$$

for the homotopy orbit complex, so instead we will usually abbreviate it as

$$\mathcal{E}(Y) = W \otimes_\pi (Y \otimes Y).$$

Similarly, when discussing results related to cosimplicial $(n+1)$-fold loop spaces, we will write

$$\mathcal{E}^n(Y) = \mathrm{sk}_n\, W \otimes_\pi (Y \otimes Y).$$

We will also overload this notation and write, for a cosimplicial space X,

$$\mathcal{E}(X) = E\pi \times_\pi (X \otimes X)$$

for the homotopy orbit cosimplicial space and

$$\mathcal{E}^n(X) = S^n \times_\pi (X \otimes X).$$

Remark (See Section 6.1)**.** The class $[y]$ is in total degree $t-s$, so we expect there to be Dyer-Lashof operations in total degrees $\geq 2(t-s)$. The vertical operations begin in total degree $2t-s$, indicating that we have missed a few. There is one other operation that we could reasonably talk about here, the one at the bottom left. Notice that if Y comes equipped with a map $W \otimes_\pi (Y \otimes Y) \to Y$, then there is a multiplication on the spectral sequence of Y. Since the bottom Dyer-Lashof operation of an element is meant to be its square, it is compelling to notice that if $[y]$ is in $E^r_{-s,t}$ then both $Q^{t-s}[y]$ and $[y]^2$ are in bidegree $(-2s, 2t)$. This may convince the skeptical reader of the validity of the shape of Figure 1.1.

1.3 Bousfield-Kan Universal Examples

For each p, define $\mathbf{D}^p_{(r,s,s)}$ as the cokernel of

$$\mathrm{sk}_{s-1}\,\Delta^p_+ \to \mathrm{sk}_{s+r-1}\,\Delta^p_+$$

in the category of simplicial based sets (here Δ^p_+ is obtained by adding a disjoint basepoint to the standard simplicial p-simplex). For $t \geq s$, define $\mathbf{D}^p_{(r,s,t)}$ by iterating the Kan suspension $t-s$ times.

$$\mathbf{D}^p_{(r,s,t)} = \Sigma^{t-s}\mathbf{D}^p_{(r,s,s)}$$

These cosimplicial spaces $\mathbf{D}^\bullet_{(r,s,t)}$ were introduced in [3] where it was shown that the (integral) homology spectral sequence has the form of Figure 1.2.

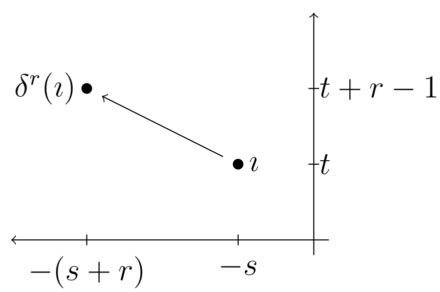

Figure 1.2. Spectral Sequence for $\mathbf{D}_{(r,s,t)}$

The Bousfield-Kan example $\mathbf{D}_{(r,s,t)}$ is universal for elements in $E^r_{-s,t}$ of the homology spectral sequence. Indeed, for a cosimplicial simplicial abelian group B and an element $b \in E^r_{-s,t}$ there is a map $\mathbb{Z}\mathbf{D}_{(r,s,t)} \to B$ which, on the spectral sequence level, sends $\imath$ to b. Slightly more general ideas can be found in [3], while slightly more specific ideas can be found in section 2.2.

In any case, the spaces $\mathbf{D}_{(r,s,t)}$ are the atomic cosimplicial spaces when it comes to the homology spectral sequence. To understand external operations, we will first understand them in these basic examples. We shall examine the spectral sequence for the cosimplicial space $E\pi \times_\pi (\mathbf{D}_{(r,s,t)} \times \mathbf{D}_{(r,s,t)})$.

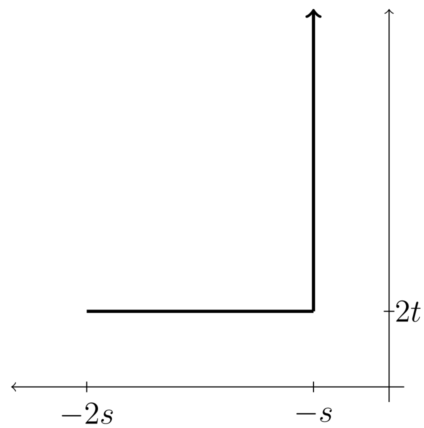

Figure 1.3. $E^2(E\pi \times_\pi (\mathbf{D}_{(\infty,s,t)} \times \mathbf{D}_{(\infty,s,t)}))$

Part of the proof of Theorem 1.0.1 we present relies on a calculation giving the extremely suggestive Figure 1.3 (or Figure 4.1 on page 35). A variation of this approach lets us replace W by its brutal truncation $\mathrm{sk}_n W$ in order to define operations in the spectral sequence of a cosimplicial E_{n+1}-space.

1.4 Outline

In Chapter 2 we apply S_* to $\mathbf{D}_{(r,s,t)}$ and compute the spectral sequence. Much of the notation used in later chapters is established here.

The next three chapters form the calculational heart of the work by providing a complete description of the homology spectral sequence of $\mathcal{E}(\mathbf{D}_{(r,s,t)})$. Chapter 3 is devoted to the calculation of E^1 and δ^1. In Chapter 4 we calculate the homology of two classes of chain complexes, which collectively give E^2. Finally, in Chapter 5 we compute E^∞ and deduce the rest of the differentials.

In Chapter 6 we will define our operations. This isn't entirely straightforward, as the universal property of the Bousfield-Kan examples does not give a unique representing map. This is where the mysterious 'w' in Theorem 1.0.1 comes from.

Chapter 7 includes the remaining calculations needed to obtain operations in the case of cosimplicial E_{n+1} spaces.

Throughout this work there will be variants of propositions and constructions which will apply to the case of cosimplicial E_{n+1}-spaces for n finite. If one is only interested in the infinite loop case, these may be safely skipped, so we have marked them with a $\bigstar$. The first example of this is Theorem$^{\bigstar}$3.1.2.

2. ALGEBRAIC BOUSFIELD-KAN EXAMPLES

In this chapter we give an explicit description of the mod-2 homology spectral sequence associated to the Bousfield-Kan universal examples.

We give two separate constructions of the E^1 page of these spectral sequences. The second, starting on page 14, gives a complete description of the spectral sequence and allows us to establish the universal property for the Bousfield-Kan universal examples. It is also relatively quick.

The first construction is more involved and only produces E^1. This relies on the observation (see [3, 3.1] or Appendix B) that, for a cosimplicial chain complex Y,

$$C(H_t(Y))^p \cong H_t(CY^p),$$

where the latter term is isomorphic to $E^1_{-p,t}(Y)$. The method we use is to first compute the levelwise homology $H_t(Y^p)$ and then calculate the (higher) coface maps $H_t(d^1), H_t(d^2), \ldots, H_t(d^{p+1})$. This will tell us about the left-hand side of the above isomorphism.

The advantage to presenting a construction along these lines is two-fold. First, it provides good practice since we will use this method of calculation later to calculate $\mathcal{E}(Y)$. More importantly, the levelwise homology of Y gives information about the levelwise homology of $\mathcal{E}(Y)$ and $\mathcal{E}^n(Y)$ (see section 3.1.1).

Most future calculations in this paper rely on the bases we choose here.

2.1 Homology of the Skeleton of the p-Simplex

Let Δ^p denote the (normalized) simplicial chains for the standard simplicial model of the p-simplex. A basis for Δ^p in dimension k is given by the set of ordered injections $[k] \hookrightarrow [p]$.

For a complex C, write $\mathrm{sk}_t(C)$ for the brutal truncation with

$$\mathrm{sk}_t(C)_k = \begin{cases} C_k & k \le t \\ 0 & k > t. \end{cases}$$

This notation is chosen because of its relation to the usual notion of simplicial skeleton: if X is a simplicial set, then

$$\mathrm{sk}_t S_* X = S_* \mathrm{sk}_t X,$$

where $S_* = N_*\mathbb{k}$ is the normalized simplicial $\mathbb{k}$-chains functor. Thus we wish to compute $\mathrm{sk}_t \Delta^p$, and since we always have

$$H_k(\mathrm{sk}_t C) = \begin{cases} 0 & k > t \\ Z_t(C) & k = t \\ H_k(C) & k < t \end{cases}$$

we are left to understand $Z_t(\Delta^p)$. If $t > 0$ then $Z_t(\Delta^p) = B_t(\Delta^p)$ and if $t = 0$ then $Z_t(\Delta^p)$ is given by the collection of vertices $\mathbb{k}\{[0] \hookrightarrow [p]\}$. In summary,

$$H_k(\mathrm{sk}_t \Delta^p) = \begin{cases} B_t(\Delta^p) & k = t > 0 \\ \mathbb{k}\{[0] \hookrightarrow [p]\} & k = t = 0 \\ \mathbb{k} & k = 0, t > k \\ 0 & \text{else.} \end{cases}$$

We now give an explicit description for $H_t(\mathrm{sk}_t \Delta^p) = B_t(\Delta^p)$ when $t > 0$.

Definition. We already have $\Delta^p_r = \mathbb{k}\{[r] \hookrightarrow [p]\}$. We'd like to consider the set of *based* injections

$$\Lambda^p_r = \{\, \varepsilon \mid \varepsilon : [r] \hookrightarrow [p], \varepsilon(0) = 0 \,\}.$$

The notation is chosen because this corresponds to the r-simplices of the 0-horn of the p-simplex.

Proposition 2.1.1. *The restriction of the differential*

$$d : \Delta^p_{t+1} \to \Delta^p_t$$

gives an isomorphism

$$\Bbbk\Lambda^p_{t+1} \xrightarrow{\cong} B_t(\Delta^p).$$

Proof. We first show that the map is injective. Let $V \subset \Delta^p_t$ have a basis consisting of $\varepsilon : [t] \hookrightarrow [p]$ with $\varepsilon(0) > 0$, which is the complement of $\Bbbk\Lambda^p_t$:

$$\Delta^p_t = V \oplus \Bbbk\Lambda^p_t.$$

Notice that the map

$$\begin{aligned} d_0 : \Bbbk\Lambda^p_{t+1} &\to \Delta^p_t \\ \varepsilon &\mapsto \varepsilon \circ d^0 \end{aligned}$$

is an injection. The following commutes

$$\begin{tikzcd} \Bbbk\Lambda^p_{t+1} \arrow[r, "d_0"] \arrow[d, "d"] & V \\ \Delta^p_t \arrow[r, equal] & V \oplus \Bbbk\Lambda^p_t \arrow[u] \end{tikzcd}$$

so $\ker d \subset \ker d_0 = 0$.

To show that d is surjective, fix a basis element $\varepsilon : [t+1] \hookrightarrow [p]$ of Δ^p_{t+1}. We need to show that $d\varepsilon$ is in the image of $d|\Bbbk\Lambda^p_{t+1}$. We treat the nontrivial case where $\varepsilon(0) > 0$ and write $\varepsilon = \bar{\varepsilon} d^0$ with $\bar{\varepsilon}(0) = 0$. Then

$$\begin{aligned} 0 &= dd\bar{\varepsilon} \\ &= d\left(\bar{\varepsilon} \sum_{l=0}^{t+2} d^l\right) \\ &= d(\varepsilon) + \sum_{l=1}^{t+2} d(\bar{\varepsilon} d^l). \end{aligned}$$

Thus we see that

$$d(\varepsilon) = d \sum_{l=1}^{t+2} \bar{\varepsilon} \circ d^l,$$

and of course $\bar{\varepsilon} d^l(0) = 0$ for $l > 0$. $\square$

Proposition 2.1.2. *The homology of the t-skeleton of the standard p-simplex is given by*

$$H_k(\operatorname{sk}_t \Delta^p) \cong \begin{cases} \Bbbk\Lambda^p_{t+1} & k = t > 0 \\ \Bbbk\{[0] \hookrightarrow [p]\} & k = t = 0 \\ \Bbbk & k = 0, t > k \\ 0 & \textit{else.} \end{cases}$$

2.2 Homology Spectral Sequence of the Bousfield-Kan Examples

Fix r, s, t and define $D^p = D^p_{rss}$ as the cokernel of the inclusion

$$\operatorname{sk}_{s-1} \Delta^p \hookrightarrow \operatorname{sk}_{s+r-1} \Delta^p.$$

The cosimplicial structure of $D^\bullet$ is induced from that of $\Delta^\bullet$. It's not hard to see that

$$\Sigma^{t-s} D_{rss} \cong N_* \Bbbk \mathbf{D}_{(r,s,t)},$$

where $\mathbf{D}_{(r,s,t)}$ is the cosimplicial space defined in [3, 5.1] and section 1.3 and N_* is the normalization functor $\Bbbk^\Delta \to \operatorname{Ch}(\Bbbk)$.

Proposition 2.2.1. *For $s \geq 0$ and $r \geq 2$, the homology of D^p_{rss} is given by*

$$H_k(D^p_{rss}) \cong \begin{cases} \Bbbk\Lambda^p_{s+r} & k = s + r - 1 \\ \Bbbk\Lambda^p_s & k = s \\ 0 & \textit{else.} \end{cases}$$

Proof. There is a short exact sequence of complexes

$$0 \to \operatorname{sk}_{s-1} \Delta^p \to \operatorname{sk}_{s+r-1} \Delta^p \to D^p \to 0$$

and, when $s \neq 1$, the result follows immediately from the associated long exact sequence and Proposition 2.1.2. When $s = 1$, the bottom map in the diagram

$$\begin{tikzcd} 0 \arrow[r] & H_1 D \arrow[r] & H_0 \operatorname{sk}_0 \arrow[r] \arrow[d, equal] & H_0 \operatorname{sk}_r \arrow[r] \arrow[d, equal] & H_0 D \arrow[r] & 0 \\ & & \Bbbk\{[0] \hookrightarrow [p]\} \arrow[r] & \Bbbk & & \end{tikzcd}$$

is surjective and $\Bbbk\Lambda_1^p$ is a p-dimensional vector space.

$\square$

Remark. The statement of this Proposition is not true for $r < 2$. We will assume that $r \geq 2$ until section 6.2, where we will momentarily need an easy calculation for $r = 1$.

We now reproduce a the mod-2 version of [3, 5.3, (i)–(iii)] using Proposition 2.2.1. Namely, we compute the E^1 page of the spectral sequence of D_{rss}. We do this using the fact (from [3, 3.1]) that

$$E^1_{-p,q}(D_{rss}) = H_q C(D_{rss})^p \cong C(H_q(D_{rss}))^p.$$

There isn't an obvious way to obtain information about the differentials from this isomorphism, so we will not prove [3, 5.3, (iv)] using this method. The answer is given in Figure 2.1 (see also Figure 2.3).

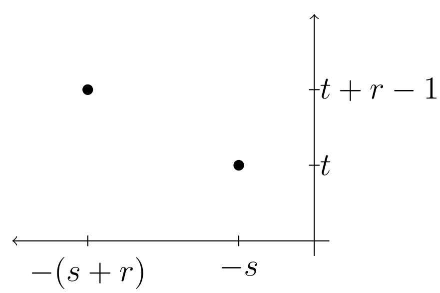

Figure 2.1. Page 1 of D_{rst}

We want to compute CHD_{rss} (noting that CHD_{rst} is obtained from this by suspension). For $j > 0$, the coface map d^j takes elements of Λ_t^p to elements of Λ_t^{p+1} (this is why I chose the 0-outer horn rather than n-outer horn: it's compatible with the convention I use for conormalization). Now if we look at the conormalization:

$$\begin{aligned}
CH(D_{rss})^p_s &= \Bbbk\Lambda^p_s / \left(d^1\Bbbk\Lambda^{p-1}_s + \cdots + d^p\Bbbk\Lambda^{p-1}_s\right)\\
&\cong \Bbbk\{\, \varepsilon \mid \varepsilon : [s] \hookrightarrow [p], \varepsilon(0) = 0, [1,p] \subset \operatorname{im}\varepsilon \,\}\\
&= \Bbbk\{\mathrm{id}_{[s]}\}\\
CH(D_{rss})^p_{s+r-1} &\cong \Bbbk\Lambda^p_{s+r} / \left(d^1\Bbbk\Lambda^{p-1}_{s+r} + \cdots + d^p\Bbbk\Lambda^{p-1}_{s+r}\right)\\
&\cong \Bbbk\{\, \varepsilon \mid \varepsilon : [s+r] \hookrightarrow [p], \varepsilon(0) = 0, [1,p] \subset \operatorname{im}\varepsilon \,\}\\
&= \Bbbk\{\mathrm{id}_{[s+r]}\}\\
CH(D_{rss})^p_k &= 0 \qquad k \neq s, s+r-1.
\end{aligned}$$

This is reflected in Figure 2.1. The separation of the two remaining classes means all intervening differentials $\delta^1, \delta^2, \ldots, \delta^{r-1}$ must be zero, so $E^1 = E^2 = \cdots = E^r$.

Instead of using Proposition 2.2.1, we could conormalize D_{rss} and get Figure 2.2. The elements on the line with y-intercept 0 are $\mathrm{id}_{[s]}, \ldots, \mathrm{id}_{[s+r-1]}$ and those on the line with y-intercept -1 are $\not{d}\,\mathrm{id}_{[s]}, \ldots, \not{d}\,\mathrm{id}_{[s+r-1]}$. One can calculate that $\not{d}\,\mathrm{id}_{[k]} = d\,\mathrm{id}_{[k+1]}$ in $C\Delta^\bullet$.

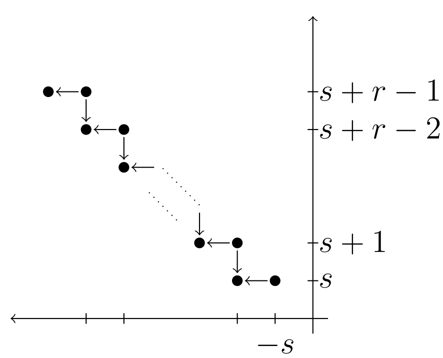

Figure 2.2. The Bicomplex $C(D_{rss})$

There is exactly one r-cycle in total degree 0, namely

$$\imath = \sum_{k=s}^{s+r-1} \mathrm{id}_{[k]}\,.$$

Clearly $\partial\imath = đ\,\mathrm{id}_{[s+r-1]}$ (or 0 if $r = \infty$). This tells us why $E^{r+1}D_{rss} = 0$ for $r < \infty$, which we did not show using the first construction. Thus, the spectral sequence is as in Figure 2.3.

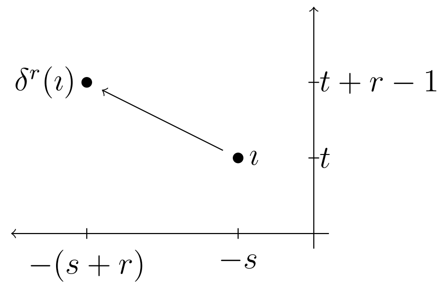

Figure 2.3. Pages 2 through r of D_{rst}

Proposition 2.2.2 (Universal Property)**.** *Let Y be a cosimplicial chain complex and $y \in Z^r_{-s,t}(Y)$. Then there is a map*

$$\Theta_y : D_{rst} \to Y$$

with

$$E^r(\Theta_y)(\imath) = [y] \qquad\qquad E^r(\Theta_y)(đ\,\mathrm{id}_{[s+r-1]}) = \delta^r[y].$$

We will be using the definition of Θ_y frequently.

Definition (Representing Map)**.** Let

$$y \in Z^r_{-s,t}(Y) \subset F^{-s}TC(Y)_{t-s}$$

which we write as

$$y = \sum_{k=0}^{\infty} y^{s+k}_{t+k} \qquad\qquad y^p_q \in C(Y)^p_q.$$

Define $C(\Theta_y)$ by

$$\Sigma^{t-s}\,\mathrm{id}_{[s+k]} \mapsto y^{s+k}_{t+k}$$
$$\Sigma^{t-s}đ\,\mathrm{id}_{[s+k]} \mapsto đy^{s+k}_{t+k}.$$

The following Proof will show that this is a map of bicomplexes, so $C(\Theta_y)$ gives Θ_y by the Dold-Kan Theorem.

Proof of Proposition 2.2.2. Since $\partial y \in F^{-s-r}$ we know that

$$\eth y_{t+k}^{s+k} = d y_{t+k+1}^{s+k+1}$$

for $0 \leq k \leq r-2$, which shows that $C(\Theta_y)$ is a map of bicomplexes. Furthermore,

$$C(\Theta_y)\imath = \sum_{k=0}^{r-1} y_{t+k}^{s+k} \sim_r \sum_{k=0}^{\infty} y_{t+k}^{s+k} = y$$

and

$$C(\Theta_y)\partial\imath = \eth y_{t+r-1}^{s+r-1}.$$

□

3. THE E^1 PAGE

3.1 Additive Structure of the E^1 Page

We are interested in the spectral sequence associated to the cosimplicial chain complex $\mathcal{E}(D_{rst})$. Note that $\mathcal{E}(D_{rst}) \cong \Sigma^{2t-2s}\mathcal{E}(D_{rss})$ where the suspension is taken levelwise, so it is enough to understand the spectral sequence for $Y = \mathcal{E}(D_{rss})$. Following [3, 3.1] (see proof in Appendix B), E^1_{-p} is isomorphic to $C(H_*Y)^p$.

Fix r and s; we now turn to computing $C(H_*Y)$ when $Y = \mathcal{E}(D_{rss})$.

3.1.1 Homology

To compute $H_*(Y)$, first notice that

$$H_*\Big(\mathcal{E}(D^p_{rss})\Big) \cong H_*\Big(\mathcal{E}(H_*(D^p_{rss}))\Big)$$

(see [4, Lemma 1.1(iii)]).

Once we have made this change, notice that, for a $\Bbbk\pi$-module M (such as $H_*(D^p)\otimes H_*(D^p)$), the complex $W \otimes_\pi M$ is just

$$\cdots M \xrightarrow{1+\sigma} M \xrightarrow{1+\sigma} M \xrightarrow{1+\sigma} M \to 0.$$

Thus the homology is $M/(1+\sigma)$ in the bottom dimension and $\ker(1+\sigma)/\operatorname{im}(1+\sigma)$ in dimensions bigger than zero. If we are working with $\operatorname{sk}_n W \otimes_\pi M$ instead, then we also get top dimensional homology $\ker(1+\sigma)$. With a little more work this gives [4, Lemma 1.3], which we now use.

We choose an order for the basis of $H_*(D^p_{rss})$ that we found in §2.2.

Definition (Total Order)**.** Every injection $\varepsilon : [m] \hookrightarrow [p]$ is given by a unique decreasing sequence $r_1 > r_2 > \cdots > r_{p-m}$ (namely, the complement of the image) with

$\varepsilon = d^{r_1} \dots d^{r_{p-m}}$. For a fixed m we declare the order on injections to be given by the reverse dictionary order on their associated sequence. For example, $d^3d^0 < d^2d^1 < d^2d^0$ and $d^3d^2 < d^3d^1$. By Proposition 2.2.1, we then have an induced order on $H_{s+r-1}(D^p)$ and on $H_s(D^p)$. We give $H_*(D^p)$ an order by declaring that $H_s(D^p) < H_{s+r-1}(D^p)$.

We apply [4, Lemma 1.3] with this totally ordered basis to see that the homology of $\mathcal{E}(H_*(D^p))$ has a basis given by the disjoint union of the following sets:

$$\{\, e_m \otimes \varepsilon \otimes \varepsilon \mid \varepsilon \in \Lambda^p_s, m \in \mathbb{N} \,\}$$
$$\{\, e_m \otimes \gamma \otimes \gamma \mid \gamma \in \Lambda^p_{s+r}, m \in \mathbb{N} \,\}$$
$$\{\, e_0 \otimes \varepsilon \otimes \varepsilon' \mid \varepsilon, \varepsilon' \in \Lambda^p_s, \varepsilon < \varepsilon' \,\}$$
$$\{\, e_0 \otimes \varepsilon \otimes \gamma \mid \varepsilon \in \Lambda^p_s, \gamma \in \Lambda^p_{s+r} \,\}$$
$$\{\, e_0 \otimes \gamma \otimes \gamma' \mid \gamma, \gamma' \in \Lambda^p_{s+r}, \gamma < \gamma' \,\}$$

To understand what happens when we apply d^k to one of these basis elements, we must use the isomorphism from Proposition 2.2.1. Notice that for $\gamma \in \Lambda^p_{s+r}$, this isomorphism is induced from d and, for $k > 0$, $d^k d\gamma = dd^k\gamma$. Thus we may use $d^k : \Bbbk\Lambda^p_{s+r} \to \Bbbk\Lambda^{p+1}_{s+r}$ for $k > 0$ without worry.

The coface maps

$$d^k : \mathcal{E}(H_*(D^p)) \to \mathcal{E}(H_*(D^{p+1}))$$

for $k > 0$ respect the basis above. One must check that $d^k\varepsilon < d^k\varepsilon'$ if $\varepsilon < \varepsilon'$. But this is easy to see. We may reduce to the case where $\varepsilon = d^{r_1} \dots d^{r_t}$ and $\varepsilon' = d^{v_1} \dots d^{v_t}$ with $r_1 > v_1$. We begin to rewrite $d^k\varepsilon$ and $d^k\varepsilon'$ in the canonical form. If $k > r_1$ then $d^k d^{r_1} \cdots$ and $d^k d^{v_1} \cdots$ is already in the canonical form, and the order is preserved. If $k \in (v_1, r_1]$ then $d^k\varepsilon = d^{r_1+1}d^k \cdots$ and $k\varepsilon' = d^k d^{v_1} \cdots$ but of course $r_1 + 1 > k$ so the order is preserved. If $k \le v_1$ then $d^k\varepsilon = d^{r_1+1} \cdots$ and $d^k\varepsilon' = d^{v_1+1} \cdots$ but of course $r_1 + 1 > v_1 + 1$.

3.1.2 Conormalization

We just saw that d^k, $k > 0$, sends basis elements in $H_*(\mathcal{E}(H_*(D^p)))$ to basis elements in $H_*(\mathcal{E}(H_*(D^{p+1})))$ via

$$e_m \otimes \varepsilon \otimes \varepsilon' \mapsto e_m \otimes d^k\varepsilon \otimes d^k\varepsilon'$$
$$\varepsilon, \varepsilon' \in \Lambda^p_s \cup \Lambda^p_{s+r} \qquad m \geq 0.$$

Thus

$$CH_*\Big(\mathcal{E}(H_*(D^p))\Big)$$

has a basis consisting of elements of the original basis which are not in the image of d^k for $k = 1, \ldots, p$. It is easy to identify such elements, which constitutes the proof of the following theorem.

Theorem 3.1.1. *Let $r \geq 2$ and $s \geq 0$. The E^1 page of the spectral sequence for the cosimplicial chain complex $\mathcal{E}(D_{rst})$ can be given a basis consisting of the following:*

In cosimplicial degree $-s$ and homological degree $2t$ and above, we have elements

$$e_m \otimes \mathrm{id}_{[s]} \otimes \mathrm{id}_{[s]} \in E^1_{-s,2t+m}$$

In cosimplicial degree $-s-r$ and homological degree $2t+2r-2$ and above, we have elements

$$e_m \otimes \mathrm{id}_{[s+r]} \otimes \mathrm{id}_{[s+r]} \in E^1_{-s-r,2t+2r+m-2}.$$

In addition, in homological degree $2t$ we have

$$e_0 \otimes \varepsilon \otimes \varepsilon' \in E^1_{-p,2t}$$

where $\varepsilon < \varepsilon' \in \Lambda^p_s$ and $[p] = \mathrm{im}\,\varepsilon \cup \mathrm{im}\,\varepsilon'$. These live in cosimplicial degrees between $-s-1$ and $-2s$.

In homological degree $2t+r-1$ we have

$$e_0 \otimes \varepsilon \otimes \gamma \in E^1_{-p,2t+r-1}$$

where $\varepsilon \in \Lambda^p_s, \gamma \in \Lambda^p_{s+r}$ *and* $[p] = \operatorname{im}\varepsilon \cup \operatorname{im}\gamma$. *These live in cosimplicial degrees between* $-s-r$ *and* $-2s-r$.

Finally, in homological degree $2t+2r-2$ *we have*

$$e_0 \otimes \gamma \otimes \gamma' \in E^1_{-p,2t+2r-2}$$

where $\gamma < \gamma' \in \Lambda^p_{s+r}$ *and* $[p] = \operatorname{im}\gamma \cup \operatorname{im}\gamma'$. *These live in cosimplicial degrees between* $-s-r-1$ *and* $-2s-2r$.

Proof. In the spectral sequence associated to $\mathcal{E}(D_{rss})$, E^1_{-p} has a basis given by the disjoint union of the following sets:

$$\{\, e_m \otimes \varepsilon \otimes \varepsilon \mid \varepsilon \in \Lambda^p_s, [1,p] \subset \operatorname{im}\varepsilon, m \in \mathbb{N} \,\}$$
$$\{\, e_m \otimes \gamma \otimes \gamma \mid \gamma \in \Lambda^p_{s+r}, [1,p] \subset \operatorname{im}\gamma, m \in \mathbb{N} \,\}$$
$$\{\, e_0 \otimes \varepsilon \otimes \varepsilon' \mid \varepsilon, \varepsilon' \in \Lambda^p_s, [1,p] \subset \operatorname{im}\varepsilon \cup \operatorname{im}\varepsilon', \varepsilon < \varepsilon' \,\}$$
$$\{\, e_0 \otimes \varepsilon \otimes \gamma \mid \varepsilon \in \Lambda^p_s, \gamma \in \Lambda^p_{s+r}, [1,p] \subset \operatorname{im}\varepsilon \cup \operatorname{im}\gamma \,\}$$
$$\{\, e_0 \otimes \gamma \otimes \gamma' \mid \gamma, \gamma' \in \Lambda^p_{s+r}, [1,p] \subset \operatorname{im}\gamma \cup \operatorname{im}\gamma', \gamma < \gamma' \,\}$$

Apply the $(2t-2s)$-fold suspension in the vertical direction to this basis. $\square$

A picture of the E^1 page is given in Figure 3.1, where we have indicated modules with rank greater than zero by snaky lines and modules of rank one with straight lines. The reader is encouraged to compare this to the picture of E^2 given in Figure 4.1 on page 35.

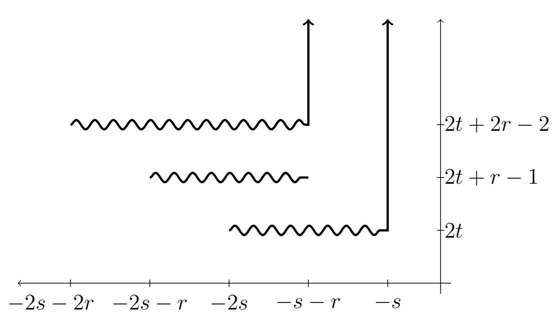

Figure 3.1. $E^1(\mathcal{E}(D_{rst}))$

We can also use [4, Lemma 1.3] to compute the homology of each of the columns when we work with the truncation of W rather than the full thing. Appropriate changes to Figure 4.2 on Page 36 will reveal the locations of the bigraded sets in the following:

Theorem★ 3.1.2. *The E^1 page of the spectral sequence associated to $\mathcal{E}^n(D_{rst})$ has a basis consisting of bigraded sets*

$$\widetilde{\mathfrak{C}}, \widetilde{\mathfrak{C}_d}, \widetilde{\mathfrak{B}}, \widetilde{\mathfrak{B}_d}, \widetilde{\mathfrak{T}}, \widetilde{\mathfrak{T}_d}, \widetilde{\mathfrak{M}_1}, \text{ and } \widetilde{\mathfrak{M}_2}.$$

(column, bottom, top, middle) We give an exhaustive list of their elements. For $m \in [0, n]$, we have

$$e_m \otimes \mathrm{id}_{[s]} \otimes \mathrm{id}_{[s]} \in \widetilde{\mathfrak{C}}_{-s,2t+m}$$

$$e_m \otimes \mathrm{id}_{[s+r]} \otimes \mathrm{id}_{[s+r]} \in (\widetilde{\mathfrak{C}_d})_{-s-r,2t+2r+m-2}.$$

For each pair $\varepsilon < \varepsilon' \in \Lambda_s^p$ and $[p] = \operatorname{im} \varepsilon \cup \operatorname{im} \varepsilon'$ we have

$$e_0 \otimes \varepsilon \otimes \varepsilon' \in \widetilde{\mathfrak{B}}_{-p,2t}$$

$$(1 + \sigma)e_n \otimes \varepsilon \otimes \varepsilon' \in \widetilde{\mathfrak{T}}_{-p,2t+n}$$

Here $-p$ is between $-s - 1$ and $-2s$.

For $\varepsilon \in \Lambda_s^p, \gamma \in \Lambda_{s+r}^p$ and $[p] = \operatorname{im} \varepsilon \cup \operatorname{im} \gamma$, we have

$$e_0 \otimes \varepsilon \otimes \gamma \in (\widetilde{\mathfrak{M}_1})_{-p,2t+r-1}$$

$$(1 + \sigma)e_n \otimes \varepsilon \otimes \gamma \in (\widetilde{\mathfrak{M}_2})_{-p,2t+r+n-1}$$

which live in degrees with $-p$ between $-s - r$ and $-2s - r$.

For $\gamma < \gamma' \in \Lambda_{s+r}^p$ and $[p] = \operatorname{im} \gamma \cup \operatorname{im} \gamma'$ we have

$$e_0 \otimes \gamma \otimes \gamma' \in (\widetilde{\mathfrak{B}_d})_{-p,2t+2r-2}$$

$$(1 + \sigma)e_n \otimes \gamma \otimes \gamma' \in (\widetilde{\mathfrak{T}_d})_{-p,2t+2r+n-2}$$

with $-p$ between $-s - r - 1$ and $-2s - 2r$.

Proof. Conormalize $H(\mathcal{E}^n(D_{rst}))$ as above. $\square$

3.2 The Differential δ^1

In Theorem 3.1.1 we gave a basis for the E^1 page of the spectral sequence of $\mathcal{E}(D_{rst})$. We now apply the cosimplicial differential $\not{d}$ to each of these basis elements. Notice that since the basis for the E^1 page is made up of *based* maps, application of $\not{d} = d^0$ *never* produces the basis elements given by Theorem 3.1.1. As Figure 3.1 indicates, only E^1-basis elements of the form $e_0 \otimes \zeta \otimes \zeta'$ may have nontrivial δ^1. This section is devoted to expressing the d-homology class of $\not{d}(e_0 \otimes \zeta \otimes \zeta')$ in terms of the basis elements of that Theorem.

Calculation of the spectral sequence does not require explicit calculation of δ^1. We include it here only for completeness.

Notation. Let $S \subset [p]$ be a set of $q+1$ elements. Define $\zeta^p(S) \in \Delta^p_q$ to be the unique ordered injection whose image is S. In other words, if $[p] - S = \{r_{p-q} > \cdots > r_1\}$ then

$$\zeta^p(S) = d^{r_{p-q}} \cdots d^{r_1} : [q] \to [p].$$

Theorem 3.2.1. *Let $S, T \subset [p]$ with $|S|, |T| \in \{s+1, s+r+1\}$, $S \cup T = [p]$, and $0 \in S \cap T$. Then in $E^1(\mathcal{E}(D_{rst}))$ we have*

$$\delta^1[e_0 \otimes \zeta^p(S) \otimes \zeta^p(T)] = \sum_{\substack{j,k \in S\cap T+1 \\ j \neq k}} [e_0 \otimes \zeta^{p+1}((S+1) \cup \{0\} - \{k\}) \otimes \zeta^{p+1}((T+1) \cup \{0\} - \{j\})].$$

Note that we stated the hypotheses in terms of subsets of $[p]$. If we instead consider injections ε and ε' with $\operatorname{im}\varepsilon = S$ and $\operatorname{im}\varepsilon' = T$ then the conditions on S and T are equivalent to $\varepsilon, \varepsilon' \in \Lambda^p_s$ and $e_0 \otimes \varepsilon \otimes \varepsilon' \neq 0$ in $C(\mathcal{E}(D))$.

This formula also holds in $E^1(\mathcal{E}^n(D_{rst}))$ and, furthermore, we have

Theorem* 3.2.2. *Let S, T be as in Theorem 3.2.1. Then in $E^1(\mathcal{E}^n(D_{rst}))$ we have*

$$\delta^1[(1+\sigma)e_n \otimes \zeta^p(S) \otimes \zeta^p(T)] = \sum_{\substack{j,k \in S\cap T+1 \\ j \neq k}} [(1+\sigma)e_n \otimes \zeta^{p+1}((S+1) \cup \{0\} - \{k\}) \otimes \zeta^{p+1}((T+1) \cup \{0\} - \{j\})].$$

This covers all differentials of elements in $\widetilde{\mathfrak{T}}$, $\widetilde{\mathfrak{M}_2}$, and $\widetilde{\mathfrak{T}_d}$. We also have the elements $[e_n \otimes \mathrm{id} \otimes \mathrm{id}]$ in $\widetilde{\mathfrak{C}}$ and $\widetilde{\mathfrak{C}_d}$ to deal with:

$$\delta^1[e_n \otimes \mathrm{id} \otimes \mathrm{id}] = \sum_{0<j<l} \left[(1+\sigma)e_n \otimes d^j \otimes d^l\right].$$

The remainder of this section is devoted to the proof of Theorem 3.2.1 and Theorem$^\bigstar$3.2.2.

The proof of Proposition 2.2.1 actually shows the following:

Lemma 3.2.3. *The map $\mathrm{sk}_{s+r-1}\Delta \to D_{rss}$ induces an isomorphism of cosimplicial modules*

$$H_{s+r-1}(\mathrm{sk}_{s+r-1}\Delta) \to H_{s+r-1}(D_{rss}).$$

For $s > 1$, the boundary map induces an isomorphism of cosimplicial modules

$$H_s(D_{rss}) \to H_{s-1}(\mathrm{sk}_{s-1}\Delta).$$

For $s = 1$ the map above $H_1(D) \to H_0(\mathrm{sk}_0\Delta)$ is an inclusion of cosimplicial modules.

□

The 'ζ'-notation makes it easy to write down d without worrying about the specific expression in terms of the d^k. We have, in D^p_{rss},

$$d(\zeta^p(S)) = \begin{cases} \sum_{k\in S} \zeta^p(S - \{k\}) & |S| \in [s+1, s+r-1] \\ 0 & \text{else.} \end{cases} \tag{3.1}$$

We can use this, as in the proof of surjectivity in Proposition 2.1.1, to show that

$$d\zeta^p(U) = \sum_{k\in U} d\zeta^p(U \cup \{0\} - \{k\})$$

in Δ when $0 \notin U$. Another elementary observation is that

$$d^0\zeta^p(S) = \zeta^{p+1}(S+1).$$

Taking $U = S+1$ these say

$$d^0 d\zeta^p(S) = dd^0\zeta^p(S) = d\zeta^{p+1}(S+1) = \sum_{k\in S+1} d\zeta^{p+1}((S+1) \cup \{0\} - \{k\}).$$

Proposition 3.2.4. *Suppose that $0 \in S \subset [p]$ and either $|S| = s+1 \geq 2$ or $|S| = s+r+1$. Then the following formula holds in $H_*(D_{rss})$:*

$$d^0(\zeta^p(S)) = \sum_{k \in S+1} \zeta^{p+1}((S+1) \cup \{0\} - \{k\}).$$

Proof. We just saw that $d^0 d\zeta^p(S) = \sum_{k \in S+1} d\zeta^{p+1}((S+1) \cup \{0\} - \{k\})$ in Δ, so the result follows from Lemma 3.2.3. $\square$

When S and T satisfy the conditions of this proposition, then, in $E^1(\mathcal{E}(D_{rst})) = CH_*(\mathcal{E}(D_{rst}))$,

$$\begin{aligned}\delta^1[e_0 \otimes \zeta^p(S) \otimes \zeta^p(T)] &= [e_0 \otimes d^0\zeta^p(S) \otimes \zeta^p(T)] \\ &= \sum_{k \in S+1} \sum_{j \in T+1} [e_0 \otimes \zeta^{p+1}((S+1) \cup \{0\} - \{k\}) \otimes \zeta^{p+1}((T+1) \cup \{0\} - \{j\})].\end{aligned}$$

Since we are working in the conormalization, if $k-1 \in S-T$, $j-1 \in T-S$, or $k=j$, then the term

$$[e_0 \otimes \zeta^{p+1}((S+1) \cup \{0\} - \{k\}) \otimes \zeta^{p+1}((T+1) \cup \{0\} - \{j\})]$$

is zero. This establishes the theorem in the cases where $|S|, |T| \geq 2$.

Remark. The preceding paragraph works if 'e_0' is replaced by '$(1+\sigma)e_n$'.

The remaining case, namely $s = 0$ with $|S| = 1$, is trivial. Notice that, for both $|T| = 1$ and $|T| = r+1$, the theorem statement reduces to the fact that $\delta^1(e_0 \otimes \zeta^p\{0\} \otimes \zeta^p(T)) = 0$. This is obvious because the target module for δ^1 is zero (see figure 3.1).

Proof of Theorem$^\bigstar$ *3.2.2.* Since the above arguments work when we replace e_0 by $(1+\sigma)e_n$, we need only to calculate

$$\delta^1[e_n \otimes \mathrm{id}_{[s']} \otimes \mathrm{id}_{[s']}] = e_n \otimes d^0 \otimes d^0,$$

where $s' = s$ or $s+r-1$. For $s' > 0$, we apply Proposition 3.2.4 to see that

$$\begin{aligned}&e_n \otimes \zeta^{s'+1}([s'+1] - \{0\}) \otimes \zeta^{s'+1}([s'+1] - \{0\}) \\ &\qquad = e_n \otimes \sum_{1 \leq j \leq s'+1} \zeta^{s'+1}([s'+1] - \{j\}) \otimes \sum_{1 \leq l \leq s'+1} \zeta^{s'+1}([s'+1] - \{l\}).\end{aligned}$$

As above, this is equal to

$$\sum_{\substack{1\leq j,l\leq s'+1\\ j\neq l}} e_n\otimes\zeta^{s'+1}([s'+1]-\{j\})\otimes\zeta^{s'+1}([s'+1]-\{l\}) = \sum_{1\leq j<l\leq s+1}(1+\sigma)e_n\otimes d^j\otimes d^l.$$

If $s=0$, then

$$\delta^1[e_n\otimes \mathrm{id}_{[0]}\otimes \mathrm{id}_{[0]}]=0$$

for bidegree reasons (since $\widetilde{\mathfrak{T}}=\emptyset$) by Theorem$^\bigstar$3.1.2.

$\square$

4. THE E^2 PAGE

The goal of this chapter is to compute the E^2 page of the spectral sequence associated to $\mathcal{E}(D_{rst})$.

We begin with some generalities on the tensor product of two cosimplicial chain complexes which we will need in section 4.2 and at various points in the rest of the paper.

The E^1 page of the spectral sequence above contains three 'horizontal strips' $[-2s,-s]\times\{2t\}$, $[-2s-r,-s-r]\times\{2t+r-1\}$, and $[-2s-2r,-s-r]\times\{2t+2r-2\}$ (if $r=\infty$ we only have the first of these) which are the only places where δ^1 may be nonzero. We introduce a slightly more general class of complexes in section 4.2 (basically including the $r=0$ and $r=1$ cases of the middle horizontal strip) to facilitate this computation, and quickly compute the cohomology of the middle horizontal strip. Then, in section 4.3 we compute the cohomology of the top and bottom strips.

4.1 Spectral Sequence of $X\otimes Y$

In this section we examine the spectral sequence associated to the tensor product of two cosimplicial chain complexes, anticipating applications in chapter 4 and section 6.1.

Let X and Y be cosimplicial chain complexes. There are two bicomplexes, $C(X)\otimes C(Y)$ and $C(X\otimes Y)$, which are readily associated to the pair. We now give natural transformations

$$C(X)\otimes C(Y)\rightleftarrows C(X\otimes Y).$$

Definition. The Alexander-Whitney map AW (see [5, p. 217] or [3, p. 316]) is defined on $C(X)^p\otimes C(Y)^q$ by

$$AW(x^p\otimes y^q)=d^{p+q}\cdots d^{p+1}x\otimes d^{p-1}\cdots d^0y.$$

The shuffle map ∇ is defined on $C(X \otimes Y)^n$ by

$$\nabla(x^n \otimes y^n) = \sum_{p+q=n} \sum_{\substack{(p,q)-\text{shuffles} \\ \tau}} s^{\tau(p)} \cdots s^{\tau(p+q-1)} x \otimes s^{\tau(0)} \cdots s^{\tau(p-1)} y$$

where we consider (p, q)-shuffles as permutations of the set

$$\{0, 1, \ldots, p + q - 1\}.$$

Lemma 4.1.1. *The Alexander-Whitney map and shuffle map are maps of bicomplexes.* □

Notice that $C(X) \otimes C(Y)$ is a retraction of $C(X \otimes Y)$:

$$\nabla \circ AW = \mathrm{id}_{C(X) \otimes C(Y)} .$$

Furthermore, if X and Y are cosimplicial abelian groups, the dual Eilenberg-Zilber theorem (see, for example, the appendix in [6]) tells us that ∇ and AW are inverse chain homotopy equivalences. In the case when X and Y are cosimplicial chain complexes, we can extend this to show that these maps give isomorphisms on E^2 (Proposition 4.1.2).

Before we begin, suppose that B and B' are bicomplexes (over $\Bbbk$), and examine the spectral sequences (obtained by filtering by columns) for $E^r(B)$, $E^r(B')$, and $E^r(B \otimes B')$. Here the tensor product

$$(B \otimes B')_{p,q} = \bigoplus_{i,j} B_{i,j} \otimes B'_{p-i,q-j}$$

is again a bicomplex. Iterated application of the Künneth isomorphism gives an isomorphism

$$E^r B \otimes E^r B' = H(E^{r-1}B) \otimes H(E^{r-1}B') \xrightarrow{\cong} H(E^{r-1}(B \otimes B')) = E^r(B \otimes B')$$

with the base case coming from $E^0 B \otimes E^0 B' = B \otimes B' = E^0(B \otimes B')$. We will generally wish to identify

$$E^r B \otimes E^r B' = E^r(B \otimes B').$$

Proposition 4.1.2. *Let X and Y be cosimplicial chain complexes. The Alexander-Whitney map*

$$C(X) \otimes C(Y) \to C(X \otimes Y)$$

induces an isomorphism

$$E^r(X) \otimes E^r(Y) \xrightarrow{\cong} E^r(X \otimes Y)$$

for all $r \geq 2$. The inverse is induced from the shuffle map ∇.

Proof. Given what came before, we really want to show that the map of bicomplexes $C(X) \otimes C(Y) \to C(X \otimes Y)$ induces a an isomorphism on page 2 of the associated spectral sequence:

$$E^2(C(X) \otimes C(Y)) \xrightarrow{\cong} E^2(C(X \otimes Y)).$$

Consider the diagram

$$\begin{tikzcd} CH_*(X) \otimes CH_*(Y) \arrow[r, "AW"] \arrow[d, equal] & C[H_*(X) \otimes H_*(Y)] \arrow[r, "\cong"] & CH_*(X \otimes Y) \arrow[d, equal] \\ H_*C(X) \otimes H_*C(Y) \arrow[r, "\cong"] & H_*[C(X) \otimes C(Y)] \arrow[r, "H_*AW"] & H_*C(X \otimes Y) \end{tikzcd}$$

where the isomorphisms come from the Künneth theorem. It is easy to see that this commutes when we consider $CH_*(X) = H_*C(X)$ as a subobject of $H_*(X)$ as in Appendix B.The dual Eilenberg-Zilber theorem implies that the top left map AW becomes an isomorphism when we take homology in the horizontal direction. Then too H_*AW is a quasi-isomorphism, implying that the composite

$$E^2(X) \otimes E^2(Y) \xrightarrow{\cong} E^2(C(X) \otimes C(Y)) \xrightarrow{E^2(AW)} E^2(X \otimes Y)$$

is an isomorphism.

Since $\nabla AW = \mathrm{id}$, the inverse map must be the one induced from ∇. ∎

Remark. One consequence of this proposition is that although

$$AW : C(Y) \otimes C(Y) \to C(Y \otimes Y)$$

is not π-equivariant, it becomes so on E^2 (see [3, Theorem 9.3(vii)]). This is because at the level of bicomplexes ∇ is π-equivariant:

$$AW\sigma = AW\sigma\,\mathrm{id} = AW\sigma(\nabla AW) = AW(\nabla\sigma)AW \sim_2 \mathrm{id}\,\sigma AW = \sigma AW.$$

4.2 Isolation of the Rows

Fix s and s' nonnegative integers and let $\Omega = \Omega_{s,s'}$ be the cochain complex

$$\Omega_{s,s'} = C(H_s(D_{\infty ss}) \otimes H_{s'}(D_{\infty s's'})).$$

When $s = s'$, $\Omega_{s,s}$ has an obvious π-action and we define

$$\bar{\Omega}_s = \Omega_{s,s}/\pi.$$

We know from Proposition 2.2.1 that a basis for $H_s(D_{\infty ss})$ is given by Λ_s^p. Let

$$\omega^p = \omega^p_{s,s'} \subset \Lambda_s^p \times \Lambda_{s'}^p$$

be the set of pairs (ζ, ζ') with $[p] = \operatorname{im}\zeta \cup \operatorname{im}\zeta'$. Following the ideas of section 3.1.2, $\omega^p_{s,s'}$ is a basis for $\Omega^p_{s,s'}$.

Remark. Observe that ω^p is nonempty exactly when $p \in [\max(s, s'), s + s']$.

Proposition 4.2.1. *Fix r, s, t, and consider the spectral sequence for $\mathcal{E}(D_{rst})$. There isomorphisms of complexes*

$$\begin{aligned}
\psi_{bot} &: \bar{\Omega}_s^p \to E^1_{-p,2t} \\
\psi_{mid} &: \Omega^p_{s,s+r} \to E^1_{-p,2t+r-1} \\
\psi_{top} &: \bar{\Omega}^p_{s+r} \to E^1_{-p,2t+2r-2}
\end{aligned}$$

Proof. Assume $t = s$. We have isomorphisms of cosimplicial modules

$$\begin{aligned}
(H_s(D_{rss}) \otimes H_s(D_{rss}))/\pi &\to H_{2s}(\mathcal{E}(H_*(D_{rss}))) \\
H_s(D_{rss}) \otimes H_{s+r-1}(D_{rss}) &\to H_{2s+r-1}(\mathcal{E}(H_*(D_{rss}))) \\
(H_{s+r-1}(D_{rss}) \otimes H_{s+r-1}(D_{rss}))/\pi &\to H_{2s+2r-2}(\mathcal{E}(H_*(D_{rss})))
\end{aligned}$$

each given by $\zeta \otimes \zeta' \mapsto e_0 \otimes \zeta \otimes \zeta'$. Applying C to the modules on the right gives the nontrivial rows of E^1.

We now identify the left hand side in the above isomorphisms. The inclusion $D_{rss} \to D_{\infty ss}$ induces an isomorphism $H_s(D_{rss}) \xrightarrow{\cong} H_s(D_{\infty ss})$. Lemma 3.2.3 gives

$$H_{s+r}(D_{\infty,s+r,s+r}) \xrightarrow[\partial]{\cong} H_{s+r-1}(\mathrm{sk}_{s+r-1}\,\Delta) \xrightarrow{\cong} H_{s+r-1}(D_{rss}).$$

Combined, these give isomorphisms

$$\begin{aligned}
\bar{\Omega}_s &\to C((H_s(D_{rss}) \otimes H_s(D_{rss}))/\pi) \\
\Omega_{s,s+r} &\to C(H_s(D_{rss}) \otimes H_{s+r-1}(D_{rss})) \\
\bar{\Omega}_{s+r,s+r} &\to C((H_{s+r-1}(D_{rss}) \otimes H_{s+r-1}(D_{rss}))/\pi).
\end{aligned}$$

□

Theorem 4.2.2. *The cohomology of* $\Omega_{s,s'}$ *is*

$$H^n\Omega_{s,s'} = \begin{cases} \Bbbk & n = s+s' \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Notice that $H_*D_{\infty ss}$ is concentrated in degree s by Proposition 2.2.1. So

$$\begin{aligned}
H^*\Omega_{s,s'} &= H^*C(H_*D_{\infty ss} \otimes H_*D_{\infty s's'}) \\
&= E^2(H_*D_{\infty ss} \otimes H_*D_{\infty s's'}) \\
&\cong E^2(H_*D_{\infty ss}) \otimes E^2(H_*D_{\infty s's'})
\end{aligned}$$

where the last isomorphism is by Proposition 4.1.2. The result follows from the computation of $E^1(D_{\infty ss})$ in section 2.2. □

4.3 Cohomology of $\bar{\Omega}$

Fix $s \geq 0$. In this section we employ the short exact sequence

$$0 \to A \to \Omega_{s,s} \to \bar{\Omega}_s \to 0$$

in order to compute the cohomology of $\bar{\Omega}_s$. We begin by identifying the complex A and studying its cohomology.

Observe that for $p > s$, if $(\zeta, \zeta') \in \omega^p_{s,s}$ then $\sigma(\zeta, \zeta') \neq (\zeta, \zeta')$. Thus $\Omega^p_{s,s}$ is a free $\Bbbk\pi$-module for $p > s$, so

$$\ker(\Omega^p_{s,s} \to \bar{\Omega}^p_s) = A^p = (1+\sigma)\Omega^p_{s,s}.$$

Furthermore, $\omega^s_{s,s} = \{(\mathrm{id}_{[s]}, \mathrm{id}_{[s]})\}$, so $A^s = 0$.

We have now identified A as the image of the map

$$\Omega_{s,s} \xrightarrow{1+\sigma} \Omega_{s,s}.$$

The kernel of this map,

$$\Upsilon = \Upsilon_s = \ker(1+\sigma : \Omega_{s,s} \to A)$$

will be of independent interest (see Proposition★4.3.3). For now, notice that $\Upsilon^p = (1+\sigma)\Omega^p = A^p$ for $p > s$ since Ω^p is $\Bbbk\pi$-free. This implies that

$$H^p\Upsilon = H^pA \qquad p \geq s+2.$$

Proposition 4.3.1. *Fix $s \geq 0$. We have*

$$H^p(A) = \begin{cases} \Bbbk & p \in [s+1, 2s], s > 0 \\ 0 & \textit{else} \end{cases} \qquad H^p(\Upsilon_s) = \begin{cases} \Bbbk & p \in [s+2, 2s], s > 1 \\ \Bbbk & p = 0 = s \\ 0 & \textit{else.} \end{cases}$$

Proof. We use the long exact cohomology sequence associated to the short exact sequence

$$0 \to \Upsilon \to \Omega \to A \to 0$$

as well as Theorem 4.2.2, which says that $H^{2s}\Omega = \Bbbk$ and $H^p\Omega = 0$ for $p \neq 2s$.

Suppose that $s \geq 2$. Examine the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H^{2s-1}A \arrow[r] & H^{2s}\Upsilon \arrow[r] & H^{2s}\Omega \arrow[r] \arrow[d, equal] & H^{2s}A \arrow[r] & 0 \\ & & & \Bbbk & & \end{tikzcd}$$

We show that $H^{2s}\Upsilon \to H^{2s}\Omega$ is zero. If $H^{2s}\Upsilon \to H^{2s}\Omega$ were nonzero, then $0 = H^{2s}A = H^{2s}\Upsilon$, which would also imply that $H^{2s}\Omega = 0$. This contradicts Theorem 4.2.2. Thus we have $\mathbb{k} = H^{2s}\Omega = H^{2s}A$, and $H^{2s-1}A \to H^{2s}\Upsilon = \mathbb{k}$ is an isomorphism. For $p < 2s$, we have that

$$0 \to H^{p-1}A \to H^p\Upsilon \to 0$$

is exact, so for $s+2 \leq p < 2s$ we have

$$H^{p-1}A = H^p\Upsilon = H^pA = \mathbb{k}.$$

To finish this case, notice that $A^{p-1} = 0$ for $p-1 \leq s$, so $0 = H^{p-1}A = H^p\Upsilon$ for $p \leq s+1$.

If $s = 1$, then $A^p = 0$ for $p \neq 2$ and $A^2 \neq 0$, so exactness of

$$\begin{tikzcd} 0 \arrow[r] & H^2\Upsilon \arrow[r] & H^2\Omega \arrow[r] \arrow[d, equal] & H^2A \arrow[r] \arrow[d, equal] & 0 \\ & & \mathbb{k} & A^2 & \end{tikzcd}$$

implies $H^2\Upsilon = 0$ and $H^2A = \mathbb{k}$.

If $s = 0$ then $A = 0$, so $\Upsilon = \Omega$ and the result is obvious.

□

Theorem 4.3.2. *The cohomology of $\bar{\Omega}_s$ is*

$$H^n\bar{\Omega}_s = \begin{cases} \mathbb{k} & n \in [s, 2s] \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We use the exact sequence

$$0 \to A \to \Omega \to \bar{\Omega} \to 0,$$

Theorem 4.2.2, and Proposition 4.3.1. Notice immediately that $H^{i-1}\bar{\Omega} \cong H^iA$ for $i < 2s$, so we are reduced to analyzing the exact sequence

$$0 \to H^{2s-1}\bar{\Omega} \to H^{2s}A \to H^{2s}\Omega \to H^{2s}\bar{\Omega} \to 0.$$

We merely need to show that $H^{2s}A \to H^{2s}\Omega$ is always zero. When $s \geq 2$, we saw at the beginning of the proof of Proposition 4.3.1 that $H^{2s}A = H^{2s}\Upsilon \to H^{2s}\Omega$ is zero. If $s = 1$, and $\alpha \in A^2 = \Upsilon^2$ is a cycle, then α is a boundary in Υ since $H^2\Upsilon_1 = 0$, hence α is a boundary in $H^2(\Omega)$. Finally, $H^0A \to H^0\Omega_{0,0}$ is trivially zero since $A = 0$ when $s = 0$. $\square$

We briefly mention the truncation case. Proposition 4.2.1 gives isomorphisms of complexes

$$\begin{aligned}
\bar{\Omega}^*_s &\to \left(\Bbbk(\widetilde{\mathfrak{B}} \sqcup \widetilde{\mathfrak{C}}_{-s,2t}), \delta^1\right) \\
\Omega^*_{s,s+r} &\to \left(\Bbbk(\widetilde{\mathfrak{M}_1}), \delta^1\right) \\
\bar{\Omega}^*_{s+r} &\to \left(\Bbbk(\widetilde{\mathfrak{B}_d} \sqcup (\widetilde{\mathfrak{C}_d})_{-s-r,2t+2r-2}), \delta^1\right)
\end{aligned}$$

For the remaining three rows, we see

Proposition$^\bigstar$ 4.3.3. *There are isomorphisms of complexes*

$$\begin{aligned}
\Upsilon^*_s &\to \left(\Bbbk(\widetilde{\mathfrak{T}} \sqcup \widetilde{\mathfrak{C}}_{-s,2t+n}), \delta^1\right) \\
\Omega^*_{s,s+r} &\to \left(\Bbbk(\widetilde{\mathfrak{M}_2}), \delta^1\right) \\
\Upsilon^*_{s+r} &\to \left(\Bbbk(\widetilde{\mathfrak{T}_d} \sqcup (\widetilde{\mathfrak{C}_d})_{-s-r,2t+2r+n-2}), \delta^1\right)
\end{aligned}$$

Proof. The map

$$\begin{aligned}
H_s(D_{rss}) \otimes H_{s+r-1}(D_{rss}) &\to H_*(\mathcal{E}^n(H_*(D_{rss}))) \\
\varepsilon \otimes \gamma' &\mapsto (1+\sigma)e_n \otimes \varepsilon \otimes \gamma'
\end{aligned}$$

induces the middle isomorphism, as in the proof of Proposition 4.2.1. Furthermore, if $M^\bullet$ is one of the cosimplicial modules

$$H_s(D_{rss}) \otimes H_s(D_{rss}) \text{ or } H_{s+r-1}(D_{rss}) \otimes H_{s+r-1}(D_{rss}),$$

then we have a cosimplicial map

$$\ker(1+\sigma : M \to M) \to H_*(\mathcal{E}^n(H_*(D_{rss})))$$
$$\zeta \otimes \zeta' \mapsto e_n \otimes \zeta \otimes \zeta'.$$

This is an inclusion by [4, Lemma 1.3], and of course remains so after conormalizing. Finally, it is easy to see that the conormalized map

$$\Upsilon \to E^1(\mathcal{E}^n(H_*(D_{rss})))$$

has the appropriate image. □

4.4 The E^2 page

We now record the E^2 page of the spectral sequence. See Figure 4.1.

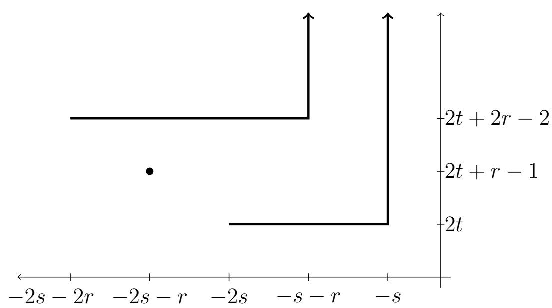

Figure 4.1. $E^2(\mathcal{E}(D_{rst}))$

Theorem 4.4.1. *For $r \in [2,\infty)$, the E^2 page of the spectral sequence for $\mathcal{E}(D_{rst})$ consists of $\Bbbk$ in the following ranges of bidegrees*

$$\{-s\} \times [2t, \infty)$$
$$\{-s-r\} \times [2t+2r-2, \infty)$$
$$[-2s, -s-1] \times \{2t\}$$
$$\{-2s-r\} \times \{2t+r-1\}$$
$$[-2s-2r, -s-r-1] \times \{2t+2r-2\}$$

and zero elsewhere.

Proof. Theorem 3.1.1 gives the E^1 page. The structure of that page gives the ranges $\{-s\} \times (2t, \infty)$ and $\{-s-r\} \times (2t+2r-2, \infty)$. Theorems 4.2.2 and 4.3.2 combine with Proposition 4.2.1 to give the rest. □

There are three possible pictures of $E^2(\mathcal{E}^n(D_{rst}))$ in the truncated case, corresponding to $n > 2r-2$, $n = 2r-2$, and $n < 2r-2$. We give the first of these in Figure 4.2, and encourage the reader to draw pictures for the other cases (using Theorem*4.4.2).

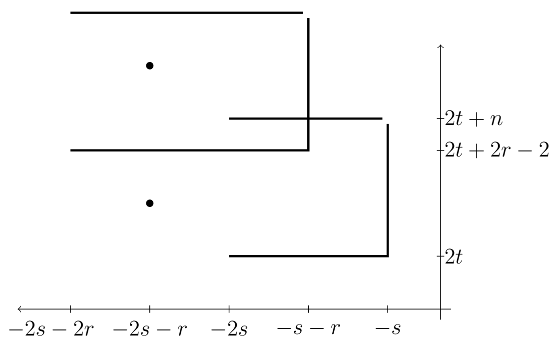

Figure 4.2. $E^2(\mathcal{E}^n(D_{rst}))$ for $n > 2r-2$

Theorem* 4.4.2. *Suppose $s > 0$. Then a basis for $E^2(\mathcal{E}^n(D_{rst}))$ is given by the union of the bigraded sets $\mathfrak{C}, \mathfrak{C}_d, \mathfrak{B}, \mathfrak{B}_d, \mathfrak{T}, \mathfrak{T}_d, \mathfrak{M}_1$, and $\mathfrak{M}_2$, each of which consists of a single element in each of the indicated bidegrees:*

$$\begin{aligned} \mathfrak{C} &: \{-s\} \times [2t, 2t+n-1] & \mathfrak{C}_d &: \{-s-r\} \times [2t+2r-2, 2t+2r+n-3] \\ \mathfrak{B} &: [-2s, -s-1] \times \{2t\} & \mathfrak{B}_d &: [-2s-2r, -s-r-1] \times \{2t+2r-2\} \\ \mathfrak{T} &: [-2s, -s-2] \times \{2t+n\} & \mathfrak{T}_d &: [-2s-2r, -s-r-2] \times \{2t+2r+n-2\} \\ \mathfrak{M}_1 &: \{(-2s-r, 2t+r-1)\} & \mathfrak{M}_2 &: \{(-2s-r, 2t+n+r-1)\} \end{aligned}$$

If $s = 0$ then the statement is the same except that $\mathfrak{C}$ is also nonempty in bidegree $(-s, 2t+n)$ (and of course $\mathfrak{B} = \emptyset = \mathfrak{T}$).

Proof. Ignoring the vertical grading, (E^1, δ^1) is the direct sum of the complexes

$$\begin{array}{ll}
\left(\Bbbk(\widetilde{\mathfrak{B}} \sqcup \widetilde{\mathfrak{C}}_{-s,2t}), \delta^1\right) & \left(\Bbbk(\widetilde{\mathfrak{B}_d} \sqcup (\widetilde{\mathfrak{C}_d})_{-s-r,2t+2r-2}), \delta^1\right) \\
\left(\Bbbk(\widetilde{\mathfrak{T}} \sqcup \widetilde{\mathfrak{C}}_{-s,2t+n}), \delta^1\right) & \left(\Bbbk(\widetilde{\mathfrak{T}_d} \sqcup (\widetilde{\mathfrak{C}_d})_{-s-r,2t+2r+n-2}), \delta^1\right) \\
\bigoplus_{i=1}^{n-1}(\Bbbk\widetilde{\mathfrak{C}}_{-s,2t+i}, 0) & \bigoplus_{i=1}^{n-1}(\Bbbk\widetilde{\mathfrak{C}_d}_{d-s-r,2t+2r-2+i}, 0) \\
\left(\Bbbk\widetilde{\mathfrak{M}_1}, \delta^1\right) & \left(\Bbbk\widetilde{\mathfrak{M}_2}, \delta^1\right).
\end{array}$$

Now combine Proposition★4.3.3 with Proposition 4.3.1. $\square$

5. PAGES E^3 THROUGH E^∞

In this chapter we complete the calculation of the spectral sequence associated to $\mathcal{E}(D_{rst})$. We begin with the easiest case – namely when $r = \infty$. Section 5.2 is devoted to algebraic convergence results. We will show that $TE^\infty(\mathcal{E}(D_{rst})) = HTC(\mathcal{E}(D_{rst}))$.

5.1 The Case $r = \infty$

We are now able to compute the spectral sequence for $\mathcal{E}(D_{\infty st})$. Proofs and statements above are generally made for finite r, but if the appropriate changes are made in §2.2 then we will end up with a variant of Theorem 3.1.1 which says

Proposition 5.1.1. *All bidegrees of $E^1(\mathcal{E}(D_{\infty st}))$ are zero except for the following:*

In cosimplicial degree $-s$ and homological degree $2t$ and above, we have elements

$$e_m \otimes \mathrm{id}_{[s]} \otimes \mathrm{id}_{[s]} \in E^1_{-s,2t+m}.$$

In homological degree $2t$ we have

$$e_0 \otimes \varepsilon \otimes \varepsilon' \in E^1_{-p,2t}$$

where $\varepsilon < \varepsilon' \in \Lambda^p_s$ and $[p] = \operatorname{im}\varepsilon \cup \operatorname{im}\varepsilon'$, which of course live in cosimplicial degree between $-2s$ and $-s-1$.

The same argument as for Theorem 4.4.1 then gives

Theorem 5.1.2. *The E^2 page of $\mathcal{E}(D_{\infty st})$ consists of $\Bbbk$ in the following bidegrees*

$$\{-s\} \times [2t, \infty)$$
$$[-2s, -s-1] \times \{2t\}$$

and zero elsewhere.

The structure of the E^2 page implies that all further differentials are zero, so

$$E^{\infty}_{-p,q} = \begin{cases} \Bbbk & q = 2t \text{ and } -p \in [-2s, -s] \\ \Bbbk & p = s \text{ and } q \geq 2t \\ 0 & \text{else.} \end{cases}$$

The following Theorem$^{\bigstar}$ can be visualized in Figure 5.1.

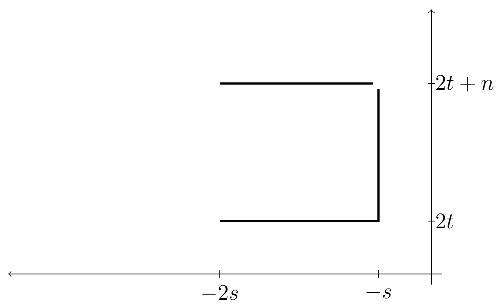

Figure 5.1. $E^2(\mathcal{E}^n(D_{\infty st}))$

Theorem$^{\bigstar}$ 5.1.3. *In the spectral sequence for $\mathcal{E}^n(D_{\infty st})$, E^2 has a basis consisting of the sets $\mathfrak{C}, \mathfrak{B}, \mathfrak{T}$ (as in Theorem$^{\bigstar}$4.4.2), or just $\mathfrak{C}$ if $s = 0$.*

Remark. $E^2 \neq E^\infty$ when $n < \infty$ and $s > 0$. There should only be $n+1$ terms on E^∞, but E^2 has $2s+n-2$ nonzero classes. We will compute the differentials in §7.1.

5.2 Algebraic Convergence

We are interested in bicomplexes arising as the conormalization of a cosimplicial chain complex, all of which are left-plane spectral sequences (the module B^p_q lies at the $(-p, q)$ lattice point). According to [7, p.142] the filtration we have defined above is complete and exhaustive.

In what follows, B is one of $C(D_{\infty st} \otimes D_{\infty st})$, or $C(\mathcal{E}^n(D_{\infty st}))$, where $n < \infty$.

Consider the short exact sequence of complexes

$$0 \to F^{-p}/F^{-p-1} \to T(B)/F^{-p-1} \to T(B)/F^{-p} \to 0.$$

For each of these examples we will show (Lemmas 5.2.1 and 5.2.2) that

$$H_m(F^{-p}/F^{-p-1}) = 0$$

for large p. This implies, of course, that

$$H_m\left(T(B)/F^{-p-1}\right) \to H_m\left(T(B)/F^{-p}\right)$$

is an isomorphism for large p, so by Mittag-Leffler

$$\varprojlim_p{}^1 H_m\left(T(B)/F^{-p}\right) = 0.$$

We then have the short exact sequence (see [7, p.142 and 5.5.5])

$$0 \to \varprojlim_p{}^1 H_{m+1}\left(T(B)/F^{-p}\right) \to H_m(T(B)) \to \varprojlim_p H_m\left(T(B)/F^{-p}\right) \to 0$$

which implies

$$H_m(T(B)) = \varprojlim_p H_m\left(T(B)/F^{-p}\right)$$

for all m.

As a first example consider

Lemma 5.2.1. *Let $B = C(D_{\infty st} \otimes D_{\infty st})$. Then*

$$H_m F^{-p}/F^{-p-1} = 0$$

for $p \neq 2t - m$.

Proof. The Künneth Theorem gives

$$H_*(D^p \otimes D^p) = H_*(D^p) \otimes H_*(D^p)$$

which, by Proposition 2.2.1, is only nonzero for $* = 2t$. Then

$$\begin{aligned} H_m F^{-p}/F^{-p-1} &= H_{m+p} C(D \otimes D)^p \\ &= C H_{m+p}(D \otimes D)^p \\ &= 0 \qquad\qquad \text{for } m + p \neq 2t \end{aligned}$$

□

This is subsumed by the following Lemma:

Lemma 5.2.2. *Let $B = C(\mathcal{E}^n(D_{rst}))$ for $2 \leq r \leq \infty$ and $n \leq \infty$. If $r < \infty$ let $j = 2s + 2r$ and if $r = \infty$ let $j = 2s$. Then*

$$H_m(F^{-p}/F^{-p-1}) = 0$$

for $p > j$.

Proof. This is a straightforward computation using Theorem 3.1.1 (see figure on page 35) or Theorem★3.1.2 (figure on page 36) in the case r is finite and Proposition 5.1.1 or its truncated variant (figure on page 40) in the case $r = \infty$.

$$\begin{aligned} H_m F^{-p}/F^{-p-1} &= H_{m+p} C(\mathcal{E}^n(D_{\infty st}))^p \\ &= C H_{m+p}(\mathcal{E}^n(D_{\infty st}))^p \\ &= 0 && \text{for } p > j \end{aligned}$$

□

We also have regularity of the spectral sequence since each group $E^1_{-s,t}$ is a finite $\Bbbk$-module.

Theorem 5.2.3. *Let $2 \leq r \leq \infty$ and $0 \leq n \leq \infty$. Then*

$$H_* TC(\mathcal{E}^n(D_{rst})) \cong TE^\infty(\mathcal{E}^n(D_{rst})).$$

Proof. This follows from Lemma 5.2.2 and regularity. □

5.3 $E^\infty = 0$ when $r < \infty$

Let $D = D_{rss}$ for $r < \infty$. The goal of this section is contained in its title: we wish to show that

$$E^\infty(\mathcal{E}(D)) = 0.$$

Lemma 5.3.1. *If r is finite, then the bicomplex $C(D \otimes D)$ is finite.*

Proof. The vector space $C(D \otimes D)^p_m$ has a basis consisting of elements $\varepsilon \otimes \varepsilon'$ where $\varepsilon : [q] \hookrightarrow [p]$, $\varepsilon' : [q'] \hookrightarrow [p]$, $q + q' = m$, and $[1, p] \subset \operatorname{im} \varepsilon \cup \operatorname{im} \varepsilon'$. Furthermore, since we're working in D_{rss} we require that $q, q' \in [s, s + r - 1]$. Thus we see that C^p_m is zero unless $m \in [2t, 2(t + r - 1)]$ and $p \in [s, 2(s + r)]$, so $C(D \otimes D)$ is bounded. Furthermore, each C^p_m is finite. □

Proposition 5.3.2. *For r finite we have $HTC(D \otimes D) = 0$.*

Proof. Lemma 5.3.1 implies convergence, so we have

$$HTC(D \otimes D) \cong TE^\infty(D \otimes D) \cong T[E^\infty(D) \otimes E^\infty(D)]$$

by Proposition 4.1.2. We saw in section 2.2 that $E^\infty(D_{rss}) = 0$ for $r < \infty$.

□

The following Proposition works over any ground ring and, in particular, gives

$$C(W \otimes_\pi (D \otimes D)) \cong W^v \otimes_\pi C(D \otimes D).$$

Proposition 5.3.3. *Let X_* be a chain complex and let X^v_* be the bicomplex which has X as its zeroth column. If $Y^\bullet_*$ is a cosimplicial chain complex then*

$$C(X_* \otimes Y^\bullet_*) \cong X^v_* \otimes C(Y^\bullet_*).$$

□

Finiteness of $C(D \otimes D)$ allows us to conclude that

$$TC(\mathcal{E}(D)) = TC(W \otimes_\pi (D \otimes D)) \cong W \otimes_\pi TC(D \otimes D).$$

Furthermore, the functor $W \otimes_\pi -$ preserves quasi-isomorphism.

Proposition 5.3.4. *Suppose that $L \to L'$ is a map of nonnegatively-graded $\Bbbk\pi$-complexes which induces an isomorphism in homology. Then*

$$H(W \otimes_\pi L) \to H(W \otimes_\pi L')$$

is an isomorphism as well.

Proof. The Künneth spectral sequence (see [8, Theorem 2.20])

$$E_2^{p,q} = \bigoplus_{s+t=q} \mathrm{Tor}^p_{\Bbbk\pi}(H^s(W), H^t(L)) \Rightarrow H(W \otimes_\pi L)$$

is a first-quadrant spectral sequence, so it converges. The map $L \to L'$ induces an isomorphism on E_2. □

Proposition 5.3.5. *For r finite we have $TE^\infty(\mathcal{E}(D_{rst})) = HTC(\mathcal{E}(D_{rst})) = 0$.*

Proof. We already saw hat $TC(\mathcal{E}(D_{rst})) \cong W \otimes_\pi TC(D_{rst} \otimes D_{rst})$, so we have $HTC(\mathcal{E}(D_{rst})) = 0$ by Propositions 5.3.2 and 5.3.4. The spectral sequence converges by Theorem 5.2.3. □

A similar proof gives

Proposition★ 5.3.6. *If r is finite then $TE^\infty(\mathcal{E}^n(D_{rst})) = HTC(\mathcal{E}^n(D_{rst})) = 0$.*

5.4 All Other Differentials Are Automatic

A spectral sequence with E^2 page of the form of Theorem 4.4.1 with $E^\infty = 0$ can only have one pattern of differential, which we give a rough picture of in Figure 5.2. We need only consider differentials $\delta^j : E^j_{p,q} \to E^j_{p-j,q+j-1}$ for $j \geq 2$.

Proposition 5.4.1. *The following differentials are nontrivial:*

$$\begin{aligned}
\delta^r : E^r_{-2s-r,2t+r-1} &\to E^r_{-2s-2r,2t+2r-2} && \\
\delta^{2r-1} : E^{2r-1}_{p,2t} &\to E^{2r-1}_{p-2r+1,2t+2r-2} && p \in [-2s, -s-1] \\
\delta^{2r-1-b} : E^{2r-1-b}_{-s,2t+b} &\to E^{2r-1-b}_{b+1-2r-s,2t+2r-2} && b \in [0, r-2] \\
\delta^r : E^r_{-s,2t+b} &\to E^r_{-s-r,2t+b+r-1} && b \in [r-1, \infty)
\end{aligned}$$

Proof. First we look at the 'top row' $[-2s-2r, -s-r] \times \{2t+2r-2\}$. All differentials δ^j out of $E^j_{p,q}$ for $(p,q) \in [-2s-2r, -s-r] \times \{2t+2r-2\}$ must be zero. We list all possibilities for differentials mapping to this row which have the potential to be nontrivial:

$$\begin{aligned}\delta^{2r-1} &: E^{2r-1}_{p,2t} \to E^{2r-1}_{p-2r+1,2t+2r-2} \\ \delta^{r} &: E^{r}_{-2s-r,2t+r-1} \to E^{r}_{-2s-2r,2t+2r-2} \\ \delta^{j} &: E^{j}_{-s,2t+2r-j-1} \to E^{j}_{-s-j,2t+2r-2}\end{aligned}$$

where

$$\begin{aligned}p \in [-2s,-s] &\leftrightsquigarrow p-2r+1 \in [-2s-2r+1,-2r-s+1] \\ j \in [r,2r-2] &\leftrightsquigarrow -s-j \in [2-s-2r,-s-r].\end{aligned}$$

But we have that

$$[-2s-2r,-s-r] = [-2s-2r+1,-2r-s+1] \sqcup \{-2s-2r\} \sqcup [2-2r-s,-s-r],$$

so each map listed above must have rank 1.

This leaves us only with the leftmost column

$$\{-s-r\} \times [2t+2r-1,\infty)$$

and part of the rightmost column $\{-s\} \times [2t+r,\infty)$ still unaccounted for. Then it is obvious that there is only one possibility:

$$\delta^r : E^r_{-s,q} \to E^r_{-s-r,q+r-1} \qquad\qquad q \in [2t+r,\infty).$$

□

Corollary 5.4.2. *Let $E^*_{*,*}$ be the spectral sequence associated with $\mathcal{E}(D_{rst})$. We record when various bidegrees become zero; they each contain a copy of $\Bbbk$ on the previous page. First for the lower right portion*

$$\begin{aligned}E^{2r}_{p,2t} &= 0 && p \in [-2s,-s] \\ E^{2t+2r-v}_{-s,v} &= 0 && v \in [2t+1,2t+r-1] \\ E^{r+1}_{-s,q} &= 0 && q \in [2t+r,\infty)\end{aligned}$$

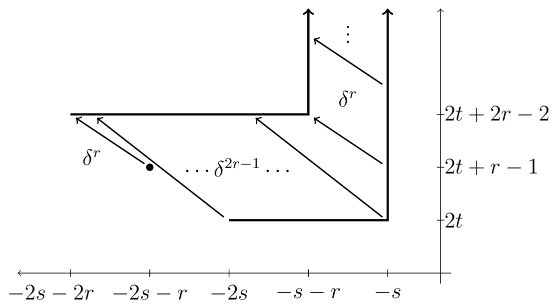

Figure 5.2. Differentials in the spectral sequence associated to $\mathcal{E}(D_{rst})$

then for the upper left portion

$$\begin{aligned} E^{r+1}_{-2s-2r,2t+2r-2} &= 0 & & \\ E^{2r}_{p,2t+2r-2} &= 0 & & p \in [-2s-2r+1, -2r-s+1] \\ E^{-s-p+1}_{p,2t+2r-2} &= 0 & & p \in [-2r-s+2, -s-r] \\ E^{r+1}_{-s-r,q} &= 0 & & q \in [2t+2r-1, \infty) \end{aligned}$$

and finally

$$E^{r+1}_{-2s-r,2t+r-1} = 0.$$

Remark. $E^{2r} = 0$.

We will not attempt to deduce the all differentials in the spectral sequence associated to $\mathcal{E}^n(D_{rst})$. Though the higher differentials are determined by the structure of E^2, the vanishing of E^∞, and naturality, even a statement along the lines of this Proposition 5.4.1 is horrendously complicated with many cases. Luckily, we will only need partial information about the differentials in this spectral sequence. We postpone further discussion of this spectral sequence until Chapter 7.

6. DEFINITION OF THE OPERATIONS

6.1 Products and Operations on Cycles

This section is a bit of a warm-up for what will come. The first goal is to define the (external) product in the spectral sequence of a cosimplicial chain complex Y and show that it is commutative. We define external operations for *r-cycles* and show that the bottom operation agrees with the external square.

In general, if Y is a cosimplicial chain complex equipped with a multiplication

$$Y \otimes Y \to Y,$$

then there is a product

$$E^r(Y) \otimes E^r(Y) \to E^r(Y)$$

which is a derivation for δ^r, coming from

$$C(Y) \otimes C(Y) \overset{AW}{\to} C(Y \otimes Y) \to C(Y).$$

In our setting, we start with a cosimplicial map

$$\theta : \mathcal{E}(Y) \to Y$$

and obtain a product by precomposition with

$$\kappa : Y \otimes Y = \Bbbk \otimes Y \otimes Y \xrightarrow{1 \mapsto e_0} W \otimes Y \otimes Y \to W \otimes_\pi (Y \otimes Y) = \mathcal{E}(Y).$$

Products $E^r(Y) \otimes E^r(Y) \to E^r(Y)$ obtained from

$$Y \otimes Y \overset{\kappa}{\to} \mathcal{E}(Y) \overset{\theta}{\to} Y$$

are commutative for $r \geq 2$.

Proposition 6.1.1. *Let Y be a cosimplicial chain complex and $r \geq 2$. The external product*

$$\mu_r : E^r(Y) \otimes E^r(Y) \xrightarrow{AW} E^r(Y \otimes Y) \to E^r(\mathcal{E}(Y))$$

is commutative.

Proof. Since AW becomes π-equivariant starting at E^2 (see the remark on page 29), we can reduce the problem to showing that the following holds on E^2:

$$\mu_2 \sigma = \kappa AW \sigma = \kappa \sigma AW \stackrel{?}{=} \kappa AW = \mu_2.$$

The equality $\mu_r \sigma = \mu_r$ then follows for all $r \geq 2$.

Thus we merely need to show that $\kappa\sigma = \kappa$ on E^2. This is actually true on E^1. Consider $v \in Z^1_{-s}(Y \otimes Y) \subset TC(Y \otimes Y)$, then we have the formula

$$\partial(e_1 \otimes v) = (1+\sigma)e_0 \otimes v + e_1 \otimes \partial v$$

in $T[W^v \otimes_\pi C(Y \otimes Y)] = TC(\mathcal{E}(Y))$. Notice that

$$\partial(e_1 \otimes v) \in \partial F^{-s} = \partial Z^0_{-s+1-1} \subset B^1_{-s}$$

and

$$e_1 \otimes \partial v \in F^{-s-1} = Z^0_{-s-1} \subset B^1_{-s}.$$

Furthermore,

$$(1+\sigma)e_0 \otimes v = (\kappa + \kappa\sigma)v,$$

so $\kappa = \kappa\sigma$ on E^1. □

We now define external operations on r-cycles using the universal property of D_{rst}. The idea is that the lower '⅃' in the spectral sequence for $\mathcal{E}(D_{rst})$ should map to the external operations. We saw in §5.4 that the 2^{nd} page is the same as the r^{th} page in this spectral sequence. Theorem 4.4.1 then says that $E^2_{p,q} = E^r_{p,q}$ is either $\Bbbk$ or 0; in the former case we write $\mathfrak{q}_{p,q}$ for the generator.

Definition. We define functions

$$\hat{Q}^m : Z^r(Y) \to E^r(\mathcal{E}(\Theta_y))$$

as follows. For $y \in Z^r_{-s,t}(Y)$, we let

$$\begin{aligned} \hat{Q}^m_v(y) &= E^r(\mathcal{E}(\Theta_y))(\mathfrak{q}_{-s,m+t}) && m \geq t \\ \hat{Q}^m_h(y) &= E^r(\mathcal{E}(\Theta_y))(\mathfrak{q}_{m-s-t,2t}) && m \in [t-s, t] \end{aligned}$$

which are all classes of $E^r(\mathcal{E}(Y))$. Here Θ_y is the map from Proposition 2.2.2.

The idea is that the lower '⅃' in the spectral sequence for $\mathcal{E}(D_{rst})$ should map to the external operations of $y \in Z^r_{-s,t}$.

Remark. Recall that on E^r, Θ_y only depends on the class of y in E^r, rather than on y itself. The situation is much more subtle for $\mathcal{E}(\Theta_y)$, and, at E^r, this map *does* depend on the specific choice of r-cycle.

Notice that an r-cycle is, in particular, an $(r-1)$-cycle. Let us now compare the answers we get by considering an r cycle in these two ways.

Proposition 6.1.2. *Let $r > 2$, and suppose $y \in Z^r_{-s,t}(Y)$. Write y_r for y considered as an element of Z^r and $y_{r-1} \in Z^{r-1}$ for y considered as an $r-1$ cycle. Then*

$$\hat{Q}^m_\bullet(y_r) = [\hat{Q}^m_\bullet(y_{r-1})]_r.$$

Before beginning the proof, notice that we can easily compare the two constructions because

$$\begin{tikzcd} D_{r-1,st} \arrow[rr, "\Theta_\imath"] \arrow[dr, "\Theta_y"'] & & D_{rst} \arrow[dl, "\Theta_y"] \\ & Y & \end{tikzcd} \tag{6.1}$$

commutes.

We will need the following Lemma. It says that if we consider the inclusion $\Theta_\imath : D_{rst} \to D_{\infty st}$, then $E^2(\mathcal{E}(\Theta_\imath))$ is an injection when restricted to the bottom '⅃'.

Lemma 6.1.3. *Consider the inclusion $\Theta_\imath : D_{rst} \to D_{\infty st}$. The map $E^2(\mathcal{E}(\Theta_\imath))$ takes $\mathfrak{q}_{p,q}$ to $\mathfrak{q}_{p,q}$ for $(p,q) \in \big([-2s,-s] \times \{2t\}\big) \cup \big(\{-s\} \times [2t,\infty)\big)$.*

Proof. On E^1, the map $\mathcal{E}(\Theta_\imath)$ is an isomorphism in this range. □

Proof of Proposition 6.1.2. A special case of diagram (6.1) is when $Y = D_{\infty st}$ and $y = \imath$. Combined with Lemma 6.1.3, this tells us that in the spectral sequence, $\mathcal{E}(D_{r-1,st}) \to \mathcal{E}(D_{rst})$ takes the lower '⅃' to the lower '⅃'. Furthermore, the following commutes,

$$\begin{tikzcd} E^2(\mathcal{E}(D_{(r-1)st})) \arrow[rr, "E^2\mathcal{E}\Theta_\imath"] \arrow[dr] & & E^2(\mathcal{E}(D_{rst})) \arrow[dl] \\ & E^2(\mathcal{E}(Y)) & \end{tikzcd}$$

which implies that the representatives on the second page of $\hat{Q}^m_\bullet(y_r)$ and $\hat{Q}^m_\bullet(y_{r-1})$ are the same. The result follows. □

6.1.1 Bottom Operation is the Square

We now show that the bottom operation coincides with the squaring operation. In particular, since the external product is commutative, this shows that the bottom operation is additive.

Lemma 6.1.4. *Let $r \geq 2$. In the case of the universal example D_{rst},*

$$\mu_r(\imath \otimes \imath) \neq 0$$

where

$$\mu_r : E^r(D_{rst}) \otimes E^r(D_{rst}) \to E^r(\mathcal{E}(D_{rst}))$$

is the external multiplication.

Proof. First notice that we can factor the external multiplication as

$$E^j(D_{rst}) \otimes E^j(D_{rst}) \xrightarrow{\cong} E^j(D_{rst} \otimes D_{rst}) \to E^j(\mathcal{E}(D_{rst})),$$

with the first arrow an isomorphism when $j \geq 2$. It is slightly easier to show that

$$E^2(D_{rst} \otimes D_{rst}) \to E^2(\mathcal{E}(D_{rst}))$$

is nontrivial; the result then follows by examining the spectral sequence for $\mathcal{E}(D_{rst})$ since nothing can hit the element in bidegree $(-2s, 2t)$.

The vertical maps in the following commutative diagram are nontrivial, where $D_{rst} \to D_{\infty st}$ is the inclusion.

$$\begin{tikzcd}
E^2_{-2s,2t}(D_{rst} \otimes D_{rst}) \arrow[r] \arrow[d] & E^2_{-2s,2t}(\mathcal{E}(D_{rst})) \arrow[d] \\
E^2_{-2s,2t}(D_{\infty st} \otimes D_{\infty st}) \arrow[r] & E^2_{-2s,2t}(\mathcal{E}(D_{\infty st}))
\end{tikzcd}$$

The diagram

$$\begin{tikzcd}
E^2_{-2s,2t}(D_{\infty st} \otimes D_{\infty st}) \arrow[r] \arrow[d, equal] & E^2_{-2s,2t}(\mathcal{E}(D_{\infty st})) \\
H^{2s}\Omega_{s,s} \arrow[r] & H^{2s}\bar{\Omega}_s \arrow[u, "\cong"', "\text{Prop. 4.2.1}"]
\end{tikzcd}$$

also commutes, and at the beginning of the proof of Theorem 4.3.2 we saw that $H^{2s}\Omega_{s,s} \to H^{2s}\bar{\Omega}_s$ is an isomorphism. □

Proposition 6.1.5. *Let $y \in Z^r_{-s,t}(Y)$. Then*

$$\mu_r([y],[y]) = \hat{Q}^{t-s}(y).$$

Proof. Let $\Theta_y : D_{rst} \to Y$ be the representing map from Proposition 2.2.2. Then

$$\hat{Q}^{t-s}(y) = E^r(\mathcal{E}(\Theta_y))(\mathfrak{q}_{-2s,2t}) = E^r(\mathcal{E}(\Theta_y))(\mu_r(\imath \otimes \imath))$$

by the preceding lemma. Since $\mu_r : E^r(-) \otimes E^r(-) \Rightarrow E^r(\mathcal{E}(-))$ is a natural transformation, we have a commutative diagram

$$\begin{tikzcd}
E^r(D_{rst}) \otimes E^r(D_{rst}) \arrow[r, "\mu_r"] \arrow[d, "E^r(\Theta_y)\otimes E^r(\Theta_y)"'] & E^r(\mathcal{E}(D_{rst})) \arrow[d, "E^r(\mathcal{E}(\Theta_y))"] \\
E^r(Y) \otimes E^r(Y) \arrow[r, "\mu_r"] & E^r(\mathcal{E}(Y))
\end{tikzcd}$$

so

$$\hat{Q}^{t-s}(y) = \mu_r(E^r(\Theta_y) \otimes E^r(\Theta_y)(\imath \otimes \imath)) = \mu_r([y],[y]).$$

□

There are two consequences to this Proposition. The first is that

$$\hat{Q}^{t-s} : Z^r_{-s,t}(Y) \to E^r(\mathcal{E}(Y))$$

is additive. This follows from commutativity of μ_r. Second, $\hat{Q}^{t-s}$ induces a homomorphism

$$\tilde{Q}^{t-s} : E^r_{-s,t}(Y) \to E^r(\mathcal{E}(Y))$$

since μ_r only depends on the E^r-class of a given r-cycle.

6.2 Additivity and Sums of Bousfield-Kan Examples

The goal of this section is to prove the following proposition for $m > t - s$.

Proposition 6.2.1 (Additivity)**.** *Let $r \geq 2$. The functions*

$$\begin{aligned} \hat{Q}^m_v :& Z^r_{-s,t}(Y) \to E^r_{-s,m+t}(\mathcal{E}(Y)) & m \geq t \\ \hat{Q}^m_h :& Z^r_{-s,t}(Y) \to E^r_{m-s-t,2t}(\mathcal{E}(Y)) & m \in [t-s, t] \end{aligned}$$

are homomorphisms.

Let $x, y \in Z^r_{-s,t}(Y)$. The following diagram commutes

$$\begin{tikzcd} D_{rst} \arrow[rr] \arrow[dr, "\Theta_{x+y}"'] & & D_{rst} \oplus D_{rst} \arrow[dl, "\Theta_x \oplus \Theta_y"] \\ & X & \end{tikzcd}$$

where the top map is the diagonal. This suggests that analyzing the spectral sequence for $\mathcal{E}(D_{rst} \oplus D_{rst})$ may be helpful in understanding additivity. We will need greater generality later, so we now investigate the spectral sequence associated to $\mathcal{E}(D_{rst} \oplus D_{r's't'})$.

If A and B are chain complexes, then

$$\Big((A \oplus B) \otimes (A \oplus B)\Big) = (A \otimes A) \oplus (B \otimes B) \oplus \big(A \otimes B \oplus B \otimes A\big)$$

as $\Bbbk\pi$-modules. Since $A \otimes B \oplus B \otimes A$ is a free π-module, we see

$$W \otimes_\pi \Big((A \oplus B) \otimes (A \oplus B)\Big)$$
$$= (W \otimes_\pi (A \otimes A)) \oplus (W \otimes_\pi (B \otimes B)) \oplus (W \otimes A \otimes B).$$

Lemma 6.2.2. *Let X and Y be cosimplicial chain complexes. Then*

$$\mathcal{E}(X \oplus Y) \cong \mathcal{E}(X) \oplus \mathcal{E}(Y) \oplus (W \otimes X \otimes Y)$$

via

$$e_n \otimes (x + y) \otimes (x' + y') \mapsto \begin{array}{c} e_n \otimes x \otimes x' + e_n \otimes y \otimes y' \\ + e_n \otimes x \otimes y' + \sigma e_n \otimes x' \otimes y \end{array}$$

and the obvious inclusions of the first two summands along with the inclusion

$$W \otimes X \otimes Y \to W \otimes_\pi \Big((X \oplus Y) \otimes (X \oplus Y)\Big)$$
$$e_n \otimes x \otimes y \mapsto e_n \otimes x \otimes y$$
$$\sigma e_n \otimes x \otimes y \mapsto \sigma e_n \otimes x \otimes y = e_n \otimes y \otimes x.$$

□

In particular,

$$\mathcal{E}(D_{rst} \oplus D_{r's't'}) \cong \mathcal{E}(D_{rst}) \oplus \mathcal{E}(D_{r's't'}) \oplus (W \otimes D_{rst} \otimes D_{r's't'}).$$

Conormalization is an additive functor, as is the functor which takes a bicomplex to its associated spectral sequence, so we see that we're left with only computing the spectral sequence for

$$W \otimes D_{rst} \otimes D_{r's't'}.$$

But this is easy – the inclusion

$$D_{rst} \otimes D_{r's't'} \to W \otimes D_{rst} \otimes D_{r's't'}$$

induces an isomorphism on E^1 by the Künneth theorem, so by Proposition 4.1.2,

$$E^j(W \otimes D_{rst} \otimes D_{r's't'}) \cong E^j(D_{rst}) \otimes E^j(D_{r's't'})$$

for $j \geq 2$. In particular, $E^2(W \otimes D_{rst} \otimes D_{r's't'})$ is zero outside of the following set of bidegrees:

$$\left\{ \begin{array}{c} (-s-s', t+t'), \\ (-s-s'-r, t+t'+r-1), (-s-s'-r', t+t'+r'-1), \\ (-s-s'-r-r', t+t'+r+r'-2). \end{array} \right\} \tag{6.2}$$

Proof of Proposition 6.2.1. Applying $\mathcal{E}$ and Lemma 6.2.2, we find

$$\begin{tikzcd}
\mathcal{E}(D_{rst}) \arrow[rr] \arrow[dr, "\mathcal{E}(\Theta_{x+y})"'] & & \mathcal{E}(D_{rst} \oplus D_{rst}) \arrow[r, equal, "6.2.2"] \arrow[dl] & \mathcal{E}(D_{rst}) \oplus \mathcal{E}(D_{rst}) \oplus (W \otimes D_{rst} \otimes D_{rst}) \arrow[dll, bend left, "\mathcal{E}(\Theta_x)+\mathcal{E}(\Theta_y)+???"] \\
& \mathcal{E}(X) & &
\end{tikzcd}$$

Using the formula from that Proposition we see that the composite

$$\mathcal{E}(D_{rst}) \to \mathcal{E}(D_{rst} \oplus D_{rst}) \twoheadrightarrow \mathcal{E}(D_{rst}) \oplus \mathcal{E}(D_{rst})$$

is just the diagonal.

We examine the map

$$E^2(\mathcal{E}(D_{rst})) \to E^2(W \otimes D_{rst} \otimes D_{rst})$$

in bidegrees $\{-s\} \times [2t, \infty)$ and $[-2s, -s-1] \times \{2t\}$. Of course $E^2(W \otimes D_{rst} \otimes D_{rst}) = E^2(D_{rst}) \otimes E^2(D_{rst})$ is zero at all of these bidegrees except for

$$E^2_{-2s,2t}(W \otimes D_{rst} \otimes D_{rst}) = \mathbb{k}.$$

In particular, we know that for $m > t - s$

$$\begin{aligned} \hat{Q}^m_v(x+y) &= \hat{Q}^m_v(x) + \hat{Q}^m_v(y) & m &\geq t \\ \hat{Q}^m_h(x+y) &= \hat{Q}^m_h(x) + \hat{Q}^m_h(y) & m &\in (t-s, t] \end{aligned}$$

We saw in the last section that $\hat{Q}^{t-s}$ is additive. $\square$

We will need the following Lemma at the beginning of the next section.

Lemma 6.2.3. *We have*

$$E^2(\mathcal{E}(D_{1st})) = 0,$$

which implies

$$E^2\left(\mathcal{E}(D_{rst} \oplus D_{1s't'})\right) \cong E^2(\mathcal{E}(D_{rst})).$$

Proof. The implication comes from

$$\mathcal{E}(D_{rst} \oplus D_{1s't'}) \cong \mathcal{E}(D_{rst}) \oplus \mathcal{E}(D_{1s't'}) \oplus (W \otimes D_{rst} \otimes D_{1s't'})$$

and the fact that $E^2(W \otimes D_{rst} \otimes D_{1s't'}) \cong E^2(D_{rst}) \otimes E^2(D_{1s't'}) = 0$.

First, notice that D_{1ss} is concentrated in homological degree s and $E^0 = E^1$ for this spectral sequence. Since

$$H_k(D^p_{1ss}) = (D^p_{1ss})_k = \begin{cases} \Delta^p_s & k = s \\ 0 & k \neq s \end{cases}.$$

The basis for the conormalization of $\mathcal{E}(D_{1ss})$ consists of

$$e_n \otimes \mathrm{id}_{[s]} \otimes \mathrm{id}_{[s]}$$
$$e_n \otimes d^0 \otimes d^0$$

and also

$$e_0 \otimes \varepsilon \otimes \varepsilon'$$

where $\varepsilon < \varepsilon'$ in Δ^p_s. Since we have the full group Δ^p_s we don't have to worry about the issues of §3.2. The differential on something in homological degree $2s$ is

$$e_0 \otimes \zeta(S) \otimes \zeta(T) \mapsto \begin{cases} e_0 \otimes \zeta(S+1) \otimes \zeta(T+1) & 0 \in S \cup T \\ 0 & 0 \notin S \cup T \end{cases}$$

(including the cases $S = T = [s]$ and $S = T = [1, s+1] \subset [s+1]$). Using the contraction

$$e_0 \otimes \zeta(S) \otimes \zeta(T) \mapsto \begin{cases} e_0 \otimes \zeta(S-1) \otimes \zeta(T-1) & 0 \notin S \cup T \\ 0 & 0 \in S \cup T \end{cases}$$

along with the fact that

$$\delta^1(e_n \otimes \mathrm{id}_{[s]} \otimes \mathrm{id}_{[s]}) = e_n \otimes d^0 \otimes d^0$$

we see that $E^2 = 0$. $\square$

6.3 Definition of the Operations

At the moment we have (additive) operations $\hat{Q}^m$ which are defined on r-cycles. The goal of this section is to show that these induce operations which are defined on classes in the spectral sequence. The simplest thing would be to show that $\hat{Q}^m$ vanishes on

$$B^r_{-s,t} = \partial Z^{r-1}_{-s+r-1,t-r+2} + Z^{r-1}_{-s-1,t+1}$$

for all m, but this does not happen. It turns out that the horizontal operations $\hat{Q}^m_h$ may be nonzero on $\partial Z^{r-1}_{-s+r-1,t-r+2}$, which leads to the indeterminacy in Theorem 6.3.6.

We begin with the easy part of $B^r_{-s,t}$: elements in lower filtration.

Lemma 6.3.1. *The homomorphisms $\hat{Q}^m$ vanish on $Z^{r-1}_{-s-1,t+1}$ for $r \geq 2$.*

Proof. Write $r' = r-1, s' = s+1, t' = t+1$ and let $y \in Z^{r'}_{-s',t'}(Y) \subset Z^r_{s,t}(Y)$. Then the following commutes

$$\begin{tikzcd} D_{rst} \arrow[rr, "\Theta_\imath"] \arrow[dr, "\Theta^r_y"'] & & D_{r's't'} \arrow[dl, "\Theta^{r'}_y"] \\ & Y & \end{tikzcd}$$

where, of course, we regard $\imath \in Z^{r'}_{-s',t'}(D_{r's't'})$ as an element of $Z^r_{-s,t}$. If $r' \geq 2$, then by Theorem 4.4.1 the E^2 page of the spectral sequence of $\mathcal{E}(D_{r's't'})$ is zero except for the following ranges of bidegrees:

$$\{-s-1\} \times [2t+2, \infty)$$

$$\{-s-r\} \times [2t+2r-2, \infty)$$

$$[-2s-2, -s-2] \times \{2t+2\}$$

$$\{-2s-r-1\} \times \{2t+r\}$$

$$[-2s-2r, -s-r-1] \times \{2t+2r-2\}$$

which means that $E^r(\mathcal{E}(D_{r's't'}))$ is zero on the ranges $\{-s\} \times [2t, \infty)$ and $[-2s, -s - 1] \times \{2t\}$ we are interested in. The diagram

$$\begin{tikzcd}
E^r(\mathcal{E}(D_{rst})) \arrow[rr] \arrow[dr] & & E^r(\mathcal{E}(D_{r's't'})) \arrow[dl] \\
& E^r(\mathcal{E}(Y)) &
\end{tikzcd}$$

commutes and the rightmost composition takes $\mathfrak{q}_{p,q}$ to zero for $(p, q) \in \{-s\}\times[2t, \infty)\cup[-2s, -s - 1] \times \{2t\}$, so all of the $\hat{Q}$ must vanish on y.

The case $r = 2$ is even easier, since we know $E^2(\mathcal{E}(D_{1s't'})) = 0$ by Lemma 6.2.3.

□

We now shift our attention to $\partial Z^{r-1}_{-s+r-1,t-r+2}$. We run into a problem immediately, for we would like to use the diagram

$$\begin{tikzcd}
D_{rst} \arrow[rr, "\Theta_{\partial\imath}"] \arrow[dr, "\Theta_{\partial y}"'] & & D_{r-1,s+1-r,r-t-2} \arrow[dl, "\Theta_y"] \\
& Y &
\end{tikzcd}$$

where $y \in Z^{r-1}_{-s+r-1,t-r+2}$, but *this diagram does not commute.* To see this, write

$$y = \sum_{j=s-r+1}^{\infty} y^j \qquad \text{where} \qquad y^j \in C(Y)^j.$$

We have

$$\partial y = \sum_{k=s}^{\infty} (\text{đ}y^{k-1} + dy^k) \in F^{-s}$$

since y is an $(r - 1)$-cycle. Then

$$C(\Theta_{\partial y}) \left(\sum_{j=s}^{s+r-1} \mathrm{id}_{[j]} \right) = \sum_{j=s}^{s+r-1} (\text{đ}y^{j-1} + dy^j)$$

and

$$\begin{aligned} C(\Theta_y)C(\Theta_{\partial\imath}) \left(\sum_{j=s}^{s+r-1} \mathrm{id}_{[j]} \right) &= C(\Theta_y)(\text{đ}\,\mathrm{id}_{[s-1]}) \\ &= \text{đ}y^{s-1}. \end{aligned}$$

Lemma 6.3.2. *If*

$$y = \sum_{j=s-r+1}^{s-1} y^j \in Z^{r-1}_{-s+r-1,t-r+2}$$

then the diagram

$$\begin{tikzcd} D_{rst} \arrow[r, "\Theta_{\partial\iota}"] \arrow[dr, "\Theta_{\partial y}"'] & D_{r-1,s+1-r,r-t-2} \arrow[d, "\Theta_y"] \\ & Y \end{tikzcd}$$

commutes.

Suppose that

$$y = \sum_{j=s-r+1}^{\infty} y^j \in Z^{r-1}_{-s+r-1,t-r+2}.$$

Then the tail

$$\sum_{j=s}^{\infty} y^j \in F^{-s} \subset Z^{r-1}_{-s+r-1,t-r+2},$$

so we can split y up into two pieces

$$y = \sum_{j=s-r+1}^{s-1} y^j + \sum_{j=s}^{\infty} y^j$$

both of which are in $Z_{-s+r-1,t-r+2}$. We treat the tail piece now, so that we can use Lemma 6.3.2 later.

Proposition 6.3.3. *The homomorphisms $\hat{Q}^m$ vanish on ∂F^{-s}.*

Proof. Lemma 6.3.1 guarantees that the operations vanish on elements of $\partial F^{-s-1} \subset Z^{r-1}_{-s-1,t+1}$. By additivity we may consider the boundary of a single element in cosimplicial degree s:

$$y = y^s_{t+1}.$$

Define a cosimplicial chain complex $\mathcal{V}$ (depending on s and t) schematically by

$$\begin{array}{ll} \Delta^\bullet_s & \text{in homological degree } t+1 \\ \big\downarrow \text{id} & \\ \Delta^\bullet_s & \text{in homological degree } t \end{array}$$

with zero outside of these two homological degrees. The conormalization of this is pictured in the right hand side of Figure 6.1, as well as a map of bicomplexes from $C(D_{rst})$ to $C(\mathcal{V})$ (open circles map to open circles).

This figure tells us that the diagram

$$\begin{tikzcd} C(D_{rst}) \arrow[rr] \arrow[dr, "\Theta_{\partial z}"'] & & C(\mathcal{V}) \arrow[dl] \\ & C(Y) & \end{tikzcd}$$

commutes, where $C(\mathcal{V}) \to C(Y)$ is the map taking the square in the former to the square

$$\begin{tikzcd} \text{đ}y^s_{t+1} \arrow[d] & y^s_{t+1} \arrow[l] \arrow[d] \\ \text{đ}dy^s_{t+1} & dy^s_{t+1} \arrow[l] \end{tikzcd}$$

in $C(Y)$. We apply $\mathcal{E}$ to get the commutative diagram

$$\begin{tikzcd} \mathcal{E}(D_{rst}) \arrow[rr] \arrow[dr, "\Theta_{\partial y}"'] & & \mathcal{E}(\mathcal{V}) \arrow[dl] \\ & \mathcal{E}(Y) & \end{tikzcd}$$

The vanishing of the vertical homology of $\mathcal{V}$ implies the vanishing of $E^1(\mathcal{E}(\mathcal{V}))$, so $E^1(\mathcal{E}(\Theta_{\partial y})) = 0$. □

Lemma 6.3.4. *The vertical maps $\hat{Q}_v$ vanish on $\partial Z^{r-1}_{-s+r-1,t-r+2}$ for $r > 2$.*

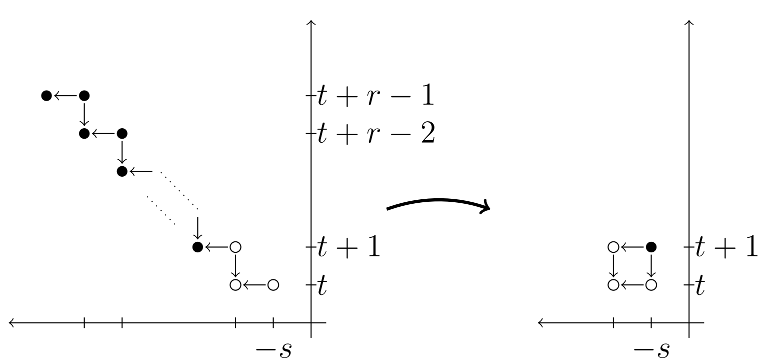

Figure 6.1. $C(D_{rst}) \to C(\mathcal{V})$

Proof. Let $r' = r-1$, $s' = s+1-r$, $t' = t-r+2$ and suppose that $y \in Z^{r-1}_{-s+r-1,t-r+2}$ has the form

$$y = \sum_{j=s-r+1}^{s-1} y^j.$$

We know this is good enough by applying the previous Proposition and additivity. Since y has this form, the following diagram commutes

$$\begin{tikzcd} D_{rst} \arrow[rr, "\Theta_{\partial\imath}"] \arrow[dr, "\Theta_{\partial y}"'] & & D_{r's't'} \arrow[dl, "\Theta_y"] \\ & Y & \end{tikzcd}$$

Applying Theorem 4.4.1 to $\mathcal{E}(D_{r's't'})$ (when $r > 2$) we find nonzero terms exactly in the following bidegrees:

$$\{-s-1-r\} \times [2t-2r+4, \infty)$$
$$\{-s\} \times [2t, \infty)$$
$$[-2s+2r-2, -s+r-2] \times \{2t-2r+4\}$$
$$\{-2s+2r-1\} \times \{2t-r+2\}$$
$$[-2s, -s-1] \times \{2t\}$$

The column $\{-s\} \times [2t, \infty)$ vanishes at E^r by the 'upper left portion' part of Corollary 5.4.2, so the vertical operations $\hat{Q}_v(\partial y)$ vanish at E^r. □

Lemma 6.3.5. *If $r = 2$ then the homomorphisms $\hat{Q}$ vanish on $\partial Z^1_{-s+1,t}$.*

Proof. Apply $E^2\mathcal{E}$ to the diagram from the proof of Lemma 6.3.4:

$$\begin{tikzcd} D_{2st} \arrow[rr, "\Theta_{\partial\imath}"] \arrow[dr, "\Theta_{\partial y}"'] & & D_{1s't'} \arrow[dl, "\Theta_y"] \\ & Y & \end{tikzcd}$$

Lemma 6.2.3 says that $E^2(\mathcal{E}(D_{1s't'})) = 0$. □

Theorem 6.3.6. *The homomorphisms of Proposition 6.2.1 induce homomorphisms*

$$\tilde{Q}^m_v : E^r_{-s,t}(X) \to E^r_{-s,m+t}(\mathcal{E}(X)) \qquad m \geq t$$
$$\tilde{Q}^m_h : E^r_{-s,t}(X) \to E^w_{m-s-t,2t}(\mathcal{E}(X)) \qquad m \in [t-s, t]$$

where

$$w = \begin{cases} r & m = t - s \\ 2r - 2 & m \in [t - s + 1, t - r + 2] \\ r + t - m & m \in [t - r + 3, t]. \end{cases}$$

Notice that if $r = 2$, then $w = 2$.

Proof. We have already shown that the vertical operations pass to this quotient using Lemmas 6.3.1 and 6.3.4. The well-definedness of the horizontal operations follows from the diagram in Lemma 6.3.4 by applying the second part of Corollary 5.4.2 to $D_{r-1,s+1-r,t-r+2}$. □

We give an example to show that the w above is the best possible. Consider the diagram

$$\begin{tikzcd} \mathcal{E}(D_{rst}) \arrow[rr, "\mathcal{E}(\Theta_{\partial\imath})"] \arrow[dr, "\mathcal{E}(\Theta_{\partial y})"'] & & \mathcal{E}(D_{r-1,s+1-r,t-r+2}) \arrow[dl, "\mathcal{E}(\Theta_y)"] \\ & \mathcal{E}(Y) & \end{tikzcd}$$

from Lemma 6.3.4.

Example. We let $t = s \geq r - 1$, and take $Y = D_{r-1,s+1-r,s-r+2}$ with

$$y = \imath = \sum_{k=s+1-r}^{k} \mathrm{id}_{[k]}$$

as our example. The following commutes

$$\begin{tikzcd} H_s(D_{rss}) \arrow[r, "\Theta_{\partial\imath}"] \arrow[d, "\cong"] & H_{s-1}(D_{r-1,s+1-r,s-r+2}) \\ H_s(D_{\infty ss}) \arrow[r, "\cong", "\partial"'] & H_{s-1}(\mathrm{sk}_{s-1}\Delta) \arrow[u, "\cong"'] \end{tikzcd}$$

and so

$$\begin{tikzcd} & \bar{\Omega}_s \arrow[dl] \arrow[dr] & \\ E^1(\mathcal{E}(D_{rss})) \arrow[rr] & & E^1(\mathcal{E}(D_{r-1,s+1-r,s-r+2})) \end{tikzcd}$$

commutes as well. Thus, at E^2, generators in the strip $[-2s, -s] \times \{2t\}$ map to nonzero elements. Vanishing of their images occurs at exactly the page described by 'w' in the Theorem.

7. COSIMPLICIAL FINITE LOOP SPACES

We now turn our attention to partial external operations. For a cosimplicial chain complex Y, these are operations whose target is the spectral sequence for

$$\mathcal{E}^n(Y) = \mathrm{sk}_n W \otimes_\pi (Y \otimes Y).$$

They are of particular interest when we have a map

$$\mathcal{E}^n(Y) \to Y,$$

such as when $Y = S_* X$ where X is a cosimplicial E_{n+1}-space.

Parallel to what we did in Chapter 6, we would like to first define operations on r-cycles:

Definition. Consider the functions

$$\hat{Q}^m : Z^r_{-s,t}(Y) \to E^r(\mathcal{E}^n(Y))$$

defined, for $y \in Z^r_{-s,t}(Y)$, by

$$\hat{Q}^m_v(y) = E^r(\mathcal{E}^n(\Theta_y))(\mathfrak{q}_{-s,m+t}) \qquad m \in [t, t-s+n]$$
$$\hat{Q}^m_h(y) = E^r(\mathcal{E}^n(\Theta_y))(\mathfrak{q}_{m-s-t,2t}) \qquad m \in [t-s, \min(t, t-s+n)]$$

where $\mathfrak{q}_{p,q} \in E^2_{p,q}(\mathcal{E}^n(D_{rst}))$ is nonzero.

Unfortunately this definition does not make any sense yet. Why should the indicated $\mathfrak{q}$ survive to page r? How do we know that there is only one generator in the indicated bidegrees? Section 7.2 is devoted to these questions and the results there show that this definition is a good one.

We expect that the top operation will not be additive, so we cannot immediately carry out the program in section 6.3 to induce operations on the spectral sequence. We will define the Browder operation in Section 7.3 in order to study the deviation from additivity of $\hat{Q}^{t-s+n}$.

7.1 Spectral Sequence Associated to $\mathcal{E}^n(D_{\infty st})$

We give a full computation of the spectral sequence indicated in the title. We do not logically need these results for what follows, but this section provides a hint that our program is reasonable, at least at the limit. Furthermore, we promised Corollary 7.1.2 way back in Section 5.1.

Theorem 7.1.1. *Consider the spectral sequence for $\mathcal{E}^n(D_{\infty st})$. The classes from E^2 which survive and are nonzero at E^∞ are exactly those of $\mathfrak{L}$ in total degrees*

$$[2t-2s, 2t-2s+n].$$

Proof. We wish (according to Theorem 5.2.3 and Proposition 5.3.3) to compute the cohomology of

$$T((\mathrm{sk}_n W)^v \otimes_\pi C(D \otimes D)),$$

where $D = D_{\infty ss}$. If we set

$$b^n_m = \dim H_m T((\mathrm{sk}_n W)^v \otimes_\pi C(D \otimes D)),$$

we already know from Theorem 5.2.3 and Section 5.1 that

$$b^\infty_m = \begin{cases} 1 & m \geq 0 \\ 0 & m < 0 \end{cases}$$

and from Theorem 5.2.3 and Proposition 4.1.2 that

$$b^0_m = \begin{cases} 1 & m = 0 \\ 0 & m \neq 0. \end{cases}$$

We use comparison and induction to interpolate between these two extremes and show that

$$b^n_m = \begin{cases} 1 & m \in [0, n] \\ 0 & \text{otherwise.} \end{cases}$$

We have two exact sequences of complexes

$$0 \to \operatorname{sk}_n W \to W \to \Sigma^{n+1} W \to 0 \tag{7.1}$$

and

$$0 \to \operatorname{sk}_0 W \to \operatorname{sk}_n W \to \Sigma \operatorname{sk}_{n-1} W \to 0 \tag{7.2}$$

where the first map is the obvious inclusion in both cases. Freeness of W_i over $\Bbbk\pi$ tells us that if we apply $(-)^v \otimes_\pi C(D \otimes D)$ to either of these exact sequences we will still have a short exact sequence of bicomplexes. Furthermore, $\prod$ is exact so applying T we again get short exact sequences of complexes – write

$$B^n = T((\operatorname{sk}_n W)^v \otimes_\pi C(D_{\infty ss} \otimes D_{\infty ss}))$$

for this composite.

We first apply $T((-)^v \otimes_\pi C(D \otimes D))$ to SES(7.2). The short exact sequence of complexes

$$0 \to B^0 \to B^n \to \Sigma B^{n-1} \to 0$$

gives us a long exact sequence in homology, so we have

$$\begin{tikzcd} H_m B^0 \arrow[r] \arrow[d, equal] & H_m B^n \arrow[r] & H_m \Sigma B^{n-1} \arrow[r] \arrow[d, equal] & H_{m-1} B^0 \arrow[d, equal] \\ 0 & & H_{m-1} B^{n-1} & 0 \end{tikzcd}$$

for $m - 1 > 0$. If we assume inductively that

$$b_m^{n-1} = \begin{cases} 1 & m \in [0, n-1] \\ 0 & \text{otherwise} \end{cases}$$

then we see that b_m^n is zero for $m \in [n+1, \infty)$ and one for $m \in [2, n]$.

Next we apply $T((-)^v \otimes_\pi C(D \otimes D))$ to SES(7.1), so we have

$$0 \to B^n \to B^\infty \to \Sigma^{n+1} B^\infty \to 0$$

and in the long exact sequence in homology we get

$$\begin{tikzcd} H_{m+1} \Sigma^{n+1} B^\infty \arrow[r] \arrow[d, equal] & H_m B^n \arrow[r] & H_m B^\infty \arrow[r] & H_m \Sigma^{n+1} B^\infty \arrow[d, equal] \\ H_{m-n} B^\infty & & & H_{m-n-1} B^\infty \end{tikzcd}$$

This tells us that $b^n_m = b^\infty_m$ for $m < n$, since there $b^\infty_{m-n} = 0 = b^\infty_{m-n-1}$.

We thus know b^n_m on $[2,\infty)$ and $(\infty, n-1]$, so for $n \geq 2$ we know it for all m. The only thing we're missing is b^1_1, but this follows from the exact sequence

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H_1B^1 & \longrightarrow & H_1\Sigma B^0 & \longrightarrow & H_0B^0 & \longrightarrow & H_0B^1 & \longrightarrow & 0 \\ & & & & \| & & \| & & \| & & \\ & & & & \Bbbk & & \Bbbk & & \Bbbk & & \end{array}$$

□

Corollary 7.1.2. *The differentials in the spectral sequence for $\mathcal{E}^n(D_{\infty st})$ are*

$$\delta^{n+1-b} : \mathfrak{C}\langle -s, 2t+b\rangle \mapsto \mathfrak{T}\langle b-s-n-1, 2t+n\rangle \qquad b \in [\max(n+1-s,0), n-1]$$
$$\delta^{n+1} : \mathfrak{B}\langle a-2s, 2t\rangle \mapsto \mathfrak{T}\langle a-2s-n-1, 2t+n\rangle \quad a \in [n+1, s-1]$$

Proof. When $s = 0$ this Corollary says that there are no nontrivial differentials, which is obvious since E^2 consists of a single column.

Assume $s > 0$ and $t = 0$. First note that $\mathfrak{L}$ lives in total degrees $[-2s, -s+n-1]$ and $\mathfrak{T}$ lives in total degrees $[-2s+n, -s+n-2]$. All differentials out of $\mathfrak{T}$ are zero, so those elements in total degrees $(-2s+n, -s+n-2]$ must be hit by something (Theorem 7.1.1). Thus we have the differentials

$$\delta^* : \mathfrak{L}(-2s+n+1, -s+n-1] \to \mathfrak{T}(-2s+n, -s+n-2]$$

are nontrivial. The classes that are unaccounted for are $\mathfrak{L}[-2s, -2s+n+1]$ and $\mathfrak{T}\{-2s+n\}$. The element in $\mathfrak{L}\{-2s+n+1\}$ doesn't survive to E^∞, and since it lives in the second page it cannot map to something in $\mathfrak{L}$. Hence there is a nontrivial differential

$$\mathfrak{L}\langle -2s+n+1\rangle \mapsto \mathfrak{T}\langle -2s+n\rangle .$$

The statement then follows by passing from total degree to bidegree. □

7.2 Spectral Sequence Associated to $\mathcal{E}^n(D_{rst})$ for $r < \infty$

We will actually need to know very little about the behavior of the spectral sequence for $\mathcal{E}^n(D_{rst})$ in order to define the operations. In this section we compute enough of the differentials to give a partial analogue to Corollary 5.4.2.

The main tool will be the comparison

$$\phi : \mathcal{E}^n(D_{rst}) \to \mathcal{E}(D_{rst})$$

induced by the inclusion

$$\operatorname{sk}_n W \hookrightarrow W.$$

Proposition 7.2.1. *Let $n \geq 1$ and $\infty > r \geq 2$. The kernel of ϕ on the second page is*

$$\ker(E^2(\phi)) = \Bbbk(\mathfrak{T} \sqcup \mathfrak{T}_d \sqcup \mathfrak{M}_2).$$

Proof. Note that the map $C(\phi)$ is just an inclusion. At E^1 the representatives of $\widetilde{\mathfrak{T}}, \widetilde{\mathfrak{T}_d}$, and $\widetilde{\mathfrak{M}_2}$ are all vertical boundaries, so $\Bbbk(\mathfrak{T} \sqcup \mathfrak{T}_d \sqcup \mathfrak{M}_2) \subset \ker(E^2(\phi))$. Comparing representatives in Theorem 3.1.1 with the representatives in Theorem$^\bigstar$3.1.2 tells us that this inclusion is equality. □

Define a set of integral lattice points $L = L^n_{rst}$ by

$$[-2s - 2r, -2s - 2r + n] \times \{2t + 2r - 2\}$$

if $n \leq s + r - 1$ and by

$$\Big([-2s - 2r, -s - r - 1] \times \{2t + 2r - 2\}\Big) \\ \cup \Big(\{-s - r\} \times [2t + 2r - 2, 2t + r - s - 2 + n]\Big)$$

if $n \geq r + s$.

Proposition 7.2.2. *If $(p, q) \in L$, then*

$$E^2_{p,q}(\mathcal{E}^n(D_{rst})) = \Bbbk.$$

Proof. We know that 1 is a lower bound for dimension since at each of these lattice points there is an element of $\mathfrak{L}_d$. The only classes at E^2 which might share a bidegree with $\mathfrak{L}_d$ (and hence with L) are $\mathfrak{T}$ and $\mathfrak{M}_2$. Notice that the lattice points of L cover the following range of total degrees:

$$[2t - 2s - 2, 2t - 2s + n - 2],$$

while by Theorem★4.4.2 we know that $\mathfrak{M}_2$ lives in total degree $2t-2s+n-1$ and $\mathfrak{T}$ lives in total degree $2t-2s+n$ and above. □

Lemma 7.2.3. *In the spectral sequence for $\mathcal{E}^n(D_{rst})$ we have*

$$\delta^r[\mathfrak{M}_1\langle -2s-r, 2t+r-1\rangle]_r = [\mathfrak{B}_d\langle -2s-2r, 2t+2r-2\rangle]_r.$$

Proof. Assume that $t=0$. Let $v \in \mathfrak{B}_d$, and $v' \in \mathfrak{M}_1$ be the elements from the statement. We list the ranges of total degrees of each of the various subsets which constitute a basis for E^2 (when $s>0$):

$$\begin{aligned} &\mathfrak{L} : [-2s, -s+n-1] && \mathfrak{L}_d : [-2s-2, -s+r+n-3] \\ &\mathfrak{T} : [-2s+n, -s+n-2] && \mathfrak{T}_d : [-2s+n-2, -s+r+n-4] \\ &\mathfrak{M}_1 : \{-2s-1\} && \mathfrak{M}_2 : \{-2s+n-1\} \end{aligned}$$

Then element v is in the smallest possible total degree $-2s-2$ so must be hit by something in total degree $-2s-1$ since $E^\infty = 0$ (Proposition★5.3.6). The only elements which are in total degree $-2s-1$ are v' and, if $n=1$, the element $\mathfrak{T}_d\langle -2s-2r, 2r-1\rangle$ (here we use that $n \geq 1$). Examination of bidegrees indicates that the second of these could only hit v via δ^1, so we have the stated result for $s>0$. The proof for $s=0$ is similar. □

Proposition 7.2.4. *Suppose that $v \in \mathfrak{L}$ has total degree in*

$$[2t-2s, 2t-2s-1+n]$$

and j is such that $\delta^j[\phi v]_j \neq 0$. Then

$$0 \neq \delta^j[v]_j \in \mathfrak{L}_d.$$

Proof. Assume $t=0$. If $[v]_j$ makes sense (that is $\delta^k[v]_k = 0$ for $k<j$), then

$$\phi\delta^j[v]_j = \delta^j\phi[v]_j = \delta^j[\phi v]_j \neq 0$$

so $\delta^j[v]_j \neq 0$. Proposition 7.2.2 coupled with Proposition 5.4.1 then tell us that it must land in the stated place.

We now show that $\delta^k[v]_k = 0$ for $2 \leq k < j$. By Lemma 7.2.3 we know that v does not hit $\mathfrak{M}_1$ nontrivially. The differential of v is in the following range of total degrees,

$$[-2s-1, -2s+n-2]$$

so we see (as in the proof of Prop. 7.2.2) that v cannot hit any of the bidegrees spanned by $\mathfrak{T}$ or $\mathfrak{M}_2$. On the other hand, $\mathfrak{T}_d$ lives in the following range of total degrees

$$[-2s+n-2, r-s+n-4],$$

so it's *possible* that v hits something in $\mathfrak{T}_d$ if it has total degree $-2s-1+n$. But $\mathfrak{T}_d$ is so far away that this must happen on a page bigger than j (see Figure 4.2 on page 36). To be precise, the differential would be one of

$$\delta^{2r+s} : \mathfrak{C}\left\langle -s, -s-1+n \right\rangle \mapsto \mathfrak{T}_d \left\langle -2s-2r, 2r+n-2 \right\rangle$$
$$\delta^{2r+n-1} : \mathfrak{B}\left\langle -2s-1+n, 0 \right\rangle \mapsto \mathfrak{T}_d \left\langle -2s-2r, 2r+n-2 \right\rangle$$

whereas $j \leq 2r-1$ by Proposition 5.4.1. □

This Proposition tells us that the spectral sequence for $\mathcal{E}^n(D_{rst})$ vanishes in the bidegrees of L at exactly the same time as in $\mathcal{E}(D_{rst})$. This is exactly what we're going to need to help us show that the $\hat{Q}^m$ vanish on appropriate boundaries.

7.3 Additivity and the Browder Operation

We would like to mimic Section 6.3, but on first glance it appears that Proposition 6.2.1 fails when $m = t-s+n$ because of the classical formula ([4, Proposition 6.5])

$$\xi_n(x+y) = \xi_n(x) + \xi_n(y) + \lambda_n(x,y)$$

where λ_n is the Browder operation and $\xi_n(x_q) = Q^{q+n}(x)$. Surprisingly, additivity holds even for the top operation as long as $s > 0$. We will see in a moment that this happens because the Browder operation lands in a lower filtration degree, but first we prove the additivity statement.

Proposition 7.3.1 (Additivity)**.** *Let $r \geq 2$ and*

$$b = \begin{cases} t - s + n & s > 0 \\ t + n - 1 & s = 0. \end{cases}$$

The functions

$$\begin{aligned} \hat{Q}_v^m &: Z^r_{-s,t}(Y) \to E^r_{-s,m+t}(\mathcal{E}^n(Y)) && m \in [t, b] \\ \hat{Q}_h^m &: Z^r_{-s,t}(Y) \to E^r_{m-s-t,2t}(\mathcal{E}^n(Y)) && m \in [t-s, \min(t, t-s+n)] \end{aligned}$$

are homomorphisms.

Proof. As in the proof of Proposition 6.2.1, we have

$$E^1(\operatorname{sk}_n W \otimes D_{rst} \otimes D_{rst}) \cong H_*(\operatorname{sk}_n W) \otimes E^1(D_{rst}) \otimes E^1(D_{rst})$$

which is nonzero only in the following list of bidegrees:

$$(-2s, 2t), (-2s - r, 2t + r - 1), (-2s - 2r, 2t + 2r - 2)$$
$$(-2s, 2t + n), (-2s - r, 2t + r - 1 + n), (-2s - 2r, 2t + 2r - 2 + n)$$

since $H_*(\operatorname{sk}_n W) = \Bbbk e_0 \oplus \Bbbk(1+\sigma)e_n$. The only possible overlap with bidegrees of the operations are $(-2s, 2t)$, which is the external square, and, if $s = 0$, $(0, 2t + n)$. But this last bidegree corresponds to the operation $\hat{Q}_v^{t+n}$ (when $s = 0$), which the one operation that is excluded from the statement of the Proposition. $\square$

Definition (Browder Operation)**.** Let Y be a cosimplicial chain complex. Consider $\Bbbk$ as a chain complex in degree 0. Using the map $\Bbbk \to \Sigma^{-n} \operatorname{sk}_n W$ which sends 1 to $(1+\sigma)e_n$ for the middle arrow below, we consider the map of bicomplexes

$$\begin{tikzcd} C(Y) \otimes C(Y) \arrow[r, "AW"] & C(Y \otimes Y) \arrow[r] & \Sigma^{-n} C(\operatorname{sk}_n W \otimes Y \otimes Y) \arrow[d] \\ & & \Sigma^{-n} C(\mathcal{E}^n(Y)) \end{tikzcd}$$

Then we have a map

$$\widetilde{\lambda}_n : E^r_{-s,t}(Y) \otimes E^r_{-s',t'}(Y) \to E^r_{-s-s',t+t'+n}(\mathcal{E}^n(Y))$$

which we call the *external Browder operation*.

There is a discrepancy in bidegrees. The top Dyer-Lashof operation for an element in $Z^r_{-s,t}$ is in bidegree $(-s, 2t+n-s)$ or $(-2s+n, 2t)$, whereas the Browder operation of two elements in $E^r_{-s,t}$ is in bidegree $(-2s, 2t+n)$. When $s = 0$ the Browder operation still measures the deviation from additivity of the top operation.

Proposition 7.3.2. *Suppose that $x, y \in Z^r_{0,t}(Y)$. Then*

$$\hat{Q}^{t+n}_v(x+y) = \hat{Q}^{t+n}_v(x) + \hat{Q}^{t+n}_v(y) + \widetilde{\lambda}_n([x]_r, [y]_r).$$

Proof. Examine the diagram from Proposition 6.2.1

$$\begin{tikzcd}
D_{r0t} \arrow[rr] \arrow[dr, "\Theta_{x+y}"'] & & D_{r0t} \oplus D_{r0t} \arrow[dl, "\Theta_x \oplus \Theta_y"] \\
& Y &
\end{tikzcd}$$

where the top map is the diagonal. Lemma 6.2.2 still works when we replace W by $\mathrm{sk}_n\, W$ and we again get the decomposition

$$\begin{tikzcd}
\mathcal{E}^n(D) \arrow[rr] \arrow[dr, "\mathcal{E}^n(\Theta_{x+y})"'] & & \mathcal{E}^n(D \oplus D) \arrow[r, equal, "6.2.2"] \arrow[dl] & \mathcal{E}^n(D) \oplus \mathcal{E}^n(D) \oplus (\mathrm{sk}_n\, W \otimes D \otimes D) \arrow[dll, bend left, "\mathcal{E}^n(\Theta_x)+\mathcal{E}^n(\Theta_y)+???"] \\
& \mathcal{E}^n(Y) & &
\end{tikzcd}$$

where $D = D_{r0t}$. The image of $\mathfrak{q}_{0,2t+n}$ under $\mathcal{E}^n(\Theta_x)$ and $\mathcal{E}^n(\Theta_y)$ give $\hat{Q}^{t+n}_v(x)$ and $\hat{Q}^{t+n}_v(y)$. We now seek to identify the image of $\mathfrak{q}_{0,2t+n}$ under the composite

$$\mathcal{E}^n(D) \to \mathrm{sk}_n\, W \otimes D \otimes D \to \mathcal{E}^n Y.$$

For maps $f : A \to C$ and $g : B \to C$, the following commutes

$$\begin{tikzcd}
\mathrm{sk}_n\, W \otimes A \otimes B \arrow[r, "1 \otimes f \otimes g"] \arrow[d] & \mathrm{sk}_n\, W \otimes C \otimes C \arrow[d] \\
\mathcal{E}^n(A \oplus B) \arrow[r, "\mathcal{E}^n(f+g)"] & \mathcal{E}^n(C)
\end{tikzcd}$$

where the left vertical arrow is the inclusion from Lemma 6.2.2. Replacing $A = B = D_{r0t}$ and $C = Y$, we extend this to the diagram

$$\begin{tikzcd}
C(D) \otimes C(D) \arrow[r, "C(\Theta_x)\otimes C(\Theta_y)"] \arrow[d, "AW"] & C(Y) \otimes C(Y) \arrow[d, "AW"] \\
C(D \otimes D) \arrow[r, "C(\Theta_x \otimes \Theta_y)"] \arrow[d] & C(Y \otimes Y) \arrow[d] \\
\Sigma^{-n} C(\mathrm{sk}_n W \otimes D \otimes D) \arrow[r, "1\otimes\Theta_x\otimes\Theta_y"] \arrow[d] & \Sigma^{-n} C(\mathrm{sk}_n W \otimes Y \otimes Y) \arrow[d] \\
\Sigma^{-n} C(\mathcal{E}^n(D \oplus D)) \arrow[r, "\mathcal{E}^n(\Theta_x+\Theta_y)"] & \Sigma^{-n} C(\mathcal{E}^n(Y))
\end{tikzcd}$$

The composite of the vertical maps on the right is what was used to define the external Browder operation, so

$$\begin{tikzcd}
E^r(D) \otimes E^r(D) \arrow[r] & E^r(Y) \otimes E^r(Y) \arrow[d] \\
& E^r(\mathcal{E}^n(Y))
\end{tikzcd}$$

takes $\imath \otimes \imath$ to $\widetilde{\lambda}_n([x]_r, [y]_r)$. Furthermore, the Alexander-Whitney map is particularly simple on elements in cosimplicial degree 0: $AW(\mathrm{id}_{[0]} \otimes \mathrm{id}_{[0]}) = \mathrm{id}_{[0]} \otimes \mathrm{id}_{[0]}$. So the vertical maps on the left give

$$\begin{array}{ccccc}
C(D) \otimes C(D) & \longrightarrow & C(D \otimes D) & \longrightarrow & \Sigma^{-n} C(\mathrm{sk}_n W \otimes D \otimes D) \\
\mathrm{id}_{[0]} \otimes \mathrm{id}_{[0]} & \longmapsto & \mathrm{id}_{[0]} \otimes \mathrm{id}_{[0]} & \longmapsto & (1+\sigma) e_n \otimes \mathrm{id}_{[0]} \otimes \mathrm{id}_{[0]} \,.
\end{array}$$

At E^1 this coincides with the image of $\mathfrak{q}_{0,2t+n}$ by Lemma 7.3.3. □

Lemma 7.3.3. *Let C be a chain complex. Consider the composite*

$$\mathcal{E}^n(C) \xrightarrow{\mathcal{E}^n \Delta} \mathcal{E}^n(C \oplus C) \twoheadrightarrow \mathrm{sk}_n W \otimes C \otimes C$$

where the projection map is essentially the one from Proposition 6.2.2:

$$\mathcal{E}^n(X \oplus Y) \to \mathrm{sk}_n W \otimes X \otimes Y$$
$$e_m \otimes (x+y) \otimes (x'+y') \mapsto e_m \otimes x \otimes y' + \sigma e_m \otimes x' \otimes y.$$

Then the homology of the composite sends $e_n \otimes [c] \otimes [c]$ to $(1+\sigma) e_n \otimes [c] \otimes [c]$.

Proof. Fix a quasi-isomorphism $C \to HC$. The following commutes,

$$\begin{tikzcd} H_*\mathcal{E}^n(C) \arrow[r] \arrow[d, "\cong"] & H_*(\mathcal{E}_n(C \oplus C)) \arrow[r] \arrow[d, "\cong"] & H_*(\mathrm{sk}_n W \otimes C \otimes C) \arrow[d, "\cong"] \\ H_*\mathcal{E}^n(HC) \arrow[r] & H^*\mathcal{E}^n(HC \oplus HC) \arrow[r] & H_*(\mathrm{sk}_n W \otimes HC \otimes HC) \end{tikzcd}$$

so it is enough to prove that for a module M,

$$H_*(\mathcal{E}^n(M)) \to H_*(\mathrm{sk}_n W \otimes M \otimes M)$$

sends $e_n \otimes m \otimes m$ to $(1+\sigma)e_n \otimes m \otimes m$. This is an easy computation. $\square$

Remark. The formula given in Proposition 7.3.2 says that if y happens to be in $B^r_{0,t}$ then

$$\hat{Q}^{t+n}_v(x+y) = \hat{Q}^{t+n}_v(x) + \hat{Q}^{t+n}_v(y)$$

since $[y]_r = 0$. So if we show that $\hat{Q}^{t+n}_v(y) = 0$ for $y \in B^r_{0,t}$ then we will know that $\hat{Q}^{t+n}_v$ induces a *function*

$$E^r_{0,t}(Y) \to E^r_{0,2t+n}(\mathcal{E}^n(Y)).$$

7.4 Definition of Operations

The proofs of nearly everything in §6.3 now go through, with perhaps the only subtle point that the analogue of Lemma 6.3.4 relies on the vanishing statement Proposition 7.2.4.

Lemma 7.4.1. *The homomorphisms $\hat{Q}^m$ vanish on $Z^{r-1}_{-s-1,t+1}$ for $r \geq 2$.*

Proof. Write $r' = r-1, s' = s+1, t' = t+1$ and let $y \in Z^{r'}_{-s',t'}(Y) \subset Z^r_{s,t}(Y)$. Then the following commutes

$$\begin{tikzcd} D_{rst} \arrow[rr, "\Theta_\imath"] \arrow[dr, "\Theta^r_y"'] & & D_{r's't'} \arrow[dl, "\Theta^{r'}_y"] \\ & X & \end{tikzcd}$$

If $r' \geq 2$, then Theorem$^\bigstar$4.4.2 says that $E^r(\mathcal{E}^n(D_{r's't'}))$ is zero on the ranges $\{-s\} \times [2t, 2t+n-s]$ and $[-2s, \min(-s-1, 2t+n-s)] \times \{2t\}$ we are interested in. The diagram

$$\begin{tikzcd} E^r(\mathcal{E}^n(D_{rst})) \arrow[rr] \arrow[dr] & & E^r(\mathcal{E}^n(D_{r's't'})) \arrow[dl] \\ & E^r(\mathcal{E}^n(Y)) & \end{tikzcd}$$

commutes and the rightmost composition takes $\mathfrak{q}_{p,q}$ to zero for (p, q) in the appropriate range, so all of the $\hat{Q}$ must vanish on x. If $r = 2$ then $E^2(\mathcal{E}^n(D_{1s't'})) = 0$. $\square$

In particular, this shows that the $\hat{Q}^m$ vanish on ∂F^{-s-1}, and the proof of the following is a minor variation of that of the corresponding Proposition in Section 6.3.

Proposition 7.4.2. *The homomorphisms $\hat{Q}^m$ vanish on ∂F^{-s}.* $\square$

Lemma 7.4.3. *The vertical maps $\hat{Q}_v$ vanish on $\partial Z^{r-1}_{-s+r-1,t-r+2}$ for $r > 2$.*

Proof. Notice that we may assume that $n \geq s$, otherwise we have not defined the vertical maps and the statement is vacuously true. Let $r' = r - 1$, $s' = s + 1 - r$, $t' = t - r + 2$. We may assume that $y \in Z^{r-1}_{-s+r-1,t-r+2}$ has the form

$$y = \sum_{j=s-r+1}^{s-1} y^j.$$

The following diagram commutes

$$\begin{tikzcd} D_{rst} \arrow[rr, "\Theta_{\partial\imath}"] \arrow[dr, "\Theta_{\partial y}"'] & & D_{r's't'} \arrow[dl, "\Theta_y"] \\ & Y & \end{tikzcd}$$

Applying Proposition 7.2.2 to (r', s', t'), we see that the vector space $E^2_{p,q}(\mathcal{E}^n D_{r's't'})$ is one-dimensional for $p = -s$ and $q \in [2t, 2t - s + n]$. Furthermore, Proposition 7.2.4 tell us that all of these classes vanish at page $r' + 1 = r$. These are exactly the bidegrees where we have defined vertical operations, so applying $E^r(\mathcal{E}^n(-))$ to the above diagram we see that $\hat{Q}_v(\partial y) = 0$ on E^r. $\square$

Lemma 7.4.4. *If $r = 2$ then the homomorphisms $\hat{Q}$ vanish on $\partial Z^1_{-s+1,t}$.*

Proof. As in Section 6.3. □

Theorem 7.4.5. *The maps above define functions*

$$\begin{aligned} \tilde{Q}^m_v :& E^r_{-s,t}(Y) \to E^r_{-s,m+t}(\mathcal{E}^n(Y)) && m \in [t, t-s+n] \\ \tilde{Q}^m_h :& E^r_{-s,t}(Y) \to E^w_{m-s-t,2t}(\mathcal{E}^n(Y)) && m \in [t-s, \min(t, t-s+n)] \end{aligned}$$

where

$$w = \begin{cases} r & m = t-s \\ 2r-2 & m \in [t-s+1, t-r+2] \\ r+t-m & m \in [t-r+3, t]. \end{cases}$$

They are homomorphisms unless $s = 0$ and $m = t-s+n$, in which case there is an error term given by Proposition 7.3.2.

Proof. The only missing ingredient is the vanishing of $\hat{Q}^m_h$ on an element ∂y where $y \in Z^{r-1}_{-s+r-1,t-r+2}$ is of the form

$$y = \sum_{j=s-r+1}^{s-1} y^j.$$

This is an extension of the proof of Lemma 7.4.3. According to Propositions 7.2.2 and 7.2.4 applied to (r', s', t'), part of Corollary 5.4.2 applies in the spectral sequence for $\mathcal{E}^n(D_{r's't'})$ to give appropriate vanishing in the range of bidegrees $[-2s+1, \min(-2s+n, -s-1)] \times \{2t\}$. In particular,

$$E^{2r-2}_{p,2t}(\mathcal{E}^n(D_{r's't'})) = 0$$

for $p \in [-2s+1, -r-s+2] \cap I$ and

$$E^{r-s-p}_{p,2t}(\mathcal{E}^n(D_{r's't'})) = 0$$

for $p \in [-r-s+3, -s] \cap I$, where $I = [-2s+1, \min(-2s+n, -s-1)]$. Furthermore, Lemma 7.2.3 tells us that

$$E^r_{-2s,2t}(\mathcal{E}^n(D_{r's't'})) = 0.$$

The statement then follows by going from p to $m = t+s+p$. □

LIST OF REFERENCES

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APPENDICES

APPENDICES

Appendix A: Operations in the Target

If X is a cosimplicial space, then the homology spectral sequence for X abuts to $H_*(\mathrm{Tot}(X))$ (see [2]). We will show in a moment that if X is a $\mathcal{C}$-space, then so is $\mathrm{Tot}(X)$. Before we do this we write down the connection between the algebraic and geometric quadratic constructions.

Proposition A.6. *If Z is a simplicial set, then there is a quasi-isomorphism*

$$\mathcal{E}(S_*Z) = W \otimes_\pi (N_*\Bbbk Z)^{\otimes 2} \to N_*\Bbbk(E\pi \times_\pi Z^{\times 2}).$$

Proof. Write $V = \Bbbk Z$ for the simplicial $\Bbbk$-module. First, the Eilenberg-Zilber theorem implies that

$$1 \otimes (\text{shuffle}) : W \otimes_\pi N_*V \otimes N_*V \to W \otimes_\pi N_*(V \otimes V)$$

is a homotopy equivalence (see [4, 7.1,7.2]), where $(V \otimes V)_p = V_p \otimes V_p = \Bbbk(Z_p \times Z_p)$.

One can show that W and $N_*\Bbbk E\pi$ are equal, or just observe that since both are $\Bbbk\pi$-free resolutions of $\Bbbk^{\mathrm{triv}}$ they are $\Bbbk\pi$-homotopy equivalent. Thus we have a homotopy equivalence

$$W \otimes_\pi N_*(V \otimes V) \to N_*\Bbbk E\pi \otimes_\pi N_*(V \otimes V).$$

Applying the Eilenberg-Zilber Theorem again (see [7, 8.5.3]) we have

$$N_*\Bbbk E\pi \otimes_{\Bbbk\pi} N_*(V \otimes V) \xrightarrow{\text{shuf}} N_*(\Bbbk E\pi \otimes_{\Bbbk\pi} V \otimes V)$$

is a quasi-isomorphism. Finally we have

$$\Bbbk E\pi_p \otimes_{\Bbbk\pi} \Bbbk Z_p \otimes \Bbbk Z_p = \Bbbk(E\pi_p \times_\pi Z_p \times Z_p)$$

coming from the standard comparison of bases:

$$\begin{tikzcd}
\text{right } \pi\text{-sets} \times \text{left } \pi\text{-sets} \arrow[r, "\Bbbk"] \arrow[d, "\times_\pi"'] & \text{right } \Bbbk\pi\text{-modules} \times \text{left } \Bbbk\pi\text{-modules} \arrow[d, "\otimes_{\Bbbk\pi}"'] \\
\text{sets} \arrow[r, "\Bbbk"] & \Bbbk\text{-modules}
\end{tikzcd}$$

□

Letting $S^n = \mathrm{sk}_n\, E\pi$, we obtain

Proposition★ A.7. *Let Z be a simplicial set. There is a natural quasi-isomorphism*

$$\mathcal{E}^n(S_*Z) \to S_*(S^n \times_\pi Z^{\times 2}).$$

Suppose that X is a cosimplicial $\mathcal{C}$-space, where $\mathcal{C}$ is an E_∞-operad. Then for each q and $H \subset \Sigma_m$, we have

$$\phi^q_m : \mathcal{C}(m) \times_H (X^q)^{\times m} \to X^q.$$

Of course for a map $[p] \to [q]$ the corresponding map $X^p \to X^q$ is a map of $\mathcal{C}$ spaces, so in particular

$$\mathcal{C}(m) \times_H (X^\bullet)^{\times m}$$

is a cosimplicial space. Furthermore, the following commutes

$$\begin{tikzcd}
\mathcal{C}(m) \times_H (X^p)^{\times m} \arrow[r] \arrow[d] & X^p \arrow[d] \\
\mathcal{C}(m) \times_H (X^q)^{\times m} \arrow[r] & X^q
\end{tikzcd}$$

commutes, so

$$\mathcal{C}(n) \times_H (X^\bullet)^{\times m} \to X^\bullet$$

is a cosimplicial map. In particular,

$$E\pi \times_\pi (X \times X) \to X$$

is a map of cosimplicial spaces.

If we totalize X, then $\mathrm{Tot}(X^\bullet) = \mathrm{Hom}_\Delta(\Delta^\bullet, X^\bullet)$ is again an $\mathcal{C}$ space. The map

$$\mathcal{C}(m) \times (\mathrm{Tot}(X))^{\times m} \to \mathrm{Tot}(X)$$

is given by

$$e \times f_1^\bullet \times f_2^\bullet \times \cdots \times f_m^\bullet \mapsto \left(t^n \mapsto \phi_m^n(e, f_1^n(t), f_2^n(t), \ldots, f_m^n(t))\right)$$

We wish to examine the triangle

$$\begin{tikzcd}
E\pi \times_\pi \mathrm{Tot}(X)^{\times 2} \arrow[rr] \arrow[dr] & & \mathrm{Tot}(X) \\
& \mathrm{Tot}(E\pi \times_\pi X^{\times 2}) \arrow[ur, "\mathrm{Tot}(\phi)"'] &
\end{tikzcd}$$

The map on the down left sends $e \times f^\bullet \times g^\bullet$ to the map $h^\bullet$ where

$$h^n(t) = e \times f^n(t) \times g^n(t).$$

Since the map going up and to the right is $\mathrm{Tot}(\phi)$, we see that this diagram commutes.

By Proposition A.6, the spectral sequence for the bottom term is isomorphic to our spectral sequence $E^*(\mathcal{E}(S_*(X)))$ on page 1 and higher. Of course by that same Proposition

$$H_*(E\pi \times_\pi \mathrm{Tot}(X)^{\times 2}) \cong H_*(\mathcal{E}(S_*(\mathrm{Tot}(X)))),$$

which is the normal target for external operations originating in $H_*(\mathrm{Tot}\, X)$. The homology of the map

$$E\pi \times_\pi \mathrm{Tot}(X)^{\times 2} \to \mathrm{Tot}(E\pi \times_\pi X^{\times 2})$$

thus takes external operations on $\mathrm{Tot}(X)$ to something in the abutment of our spectral sequence. A question that we have not yet solved is whether the external operations on $\mathrm{Tot}(X)$ map to the external operations we have defined for classes in $E^\infty(S_*(X))$.

Appendix B: $CH = HC$

Let $A^\bullet$ be a cosimplicial object in an abelian category $\mathcal{A}$. We will just write A for the (unnormalized) Moore chains. Define

$$C^p(A) = \cap_i \ker s^i : A^p \to A^{p-1}$$

and

$$D^p(A) = \sum_{i>0} d^i(A^{p-1}).$$

Proposition B.8. *The submodule $D(A)$ is a subcomplex of A.*

Proof. Fix $d^k(a)$ where $k > 0$. Then

$$d(d^k(a)) = \sum_{i\geq 0}(-1)^i d^i d^k a \equiv d^0 d^k a \mod D(A).$$

By the cosimplicial identities $d^0 d^k = d^{k+1} d^0$ and $k+1 > 0$. □

We now dualize the proof of [7, Lemma 8.3.7].

Lemma B.9.

$$D(A) \cap C(A) = 0$$

Proof. Let $y = \sum_{i>0} d^i(x_i)$. Suppose that $y \in C(A)$. If $y = 0$ there is nothing to show, so let k be the largest integer with $d^k(x_k) \neq 0$. Since $y \in C(A)$ we have $s^k(y) = 0$. A calculation then gives the equality on the right:

$$y = y - d^k s^k(y) = \sum_{0<i<k} d^i(x_i - d^{k-1}s^{k-1}x_i).$$

Induction shows that $y = 0$. □

Lemma B.10.

$$D^p(A) + C^p(A) = A^p$$

Proof. Given an element $y \in A^p$, let $m(y)$ be the smallest number so that $s^{m(y)}(y) \neq 0$, or p if $y \in C^p(A)$. It is clear that the elements in $m^{-1}(p) = C^p$ are in $D^p + C^p$. Suppose, inductively, that

$$m^{-1}(j+1) \cup m^{-1}(j+2) \cup \cdots \cup m^{-1}(p) \subset D^p + C^p.$$

We show that $m^{-1}(j) \subset D^p + C^p$. Let $y \in m^{-1}(j)$, so that $s^j(y) \neq 0$ and $s^i(y) = 0$ for $i < j$. Write

$$y' = y - d^{j+1}s^j(y) \equiv y \mod D^p.$$

Then $s^j(y') = s^j y - s^j d^{j+1} s^j y = 0$. Furthermore, for $i < j$,

$$\begin{aligned} s^i(y') &= s^i(y) - s^i d^{j+1} s^j(y) \\ &= -d^j s^i s^j(y) \\ &= -d^j s^{j-1} s^i(y) = 0. \end{aligned}$$

Thus $y' \in C^p + D^p$ by induction, so $y \in C^p + D^p$ as well. $\square$

Proposition B.11. $A = C(A) \oplus D(A)$ $\square$

One consequence of this Proposition is that CA, as we have defined it here, is isomorphic to $C(A) = A/D(A)$ from the introduction.

Let Y be a cosimplicial chain complex. By definition, $\partial^\bullet$ commutes with the cosimplicial structure maps d^k and s^i, so $\partial|_{C(Y)}$ lands in $C(Y)$ and $\partial|_{D(Y)}$ lands in $D(Y)$.

This shows that we have the decomposition

$$H_t Y^p = H_t C^p Y \oplus H_t D^p Y.$$

Of course, there is the alternate decomposition given by

$$(H_t Y)^p = C^p(H_t Y) \oplus D^p(H_t Y)$$

that we originally found. Since s^i is zero on $H^t C^p Y$, we have $H_t C^p Y \subset C^p H_t Y \subset H_t Y^p$.

It is enough to show that

$$C^p H_t Y \cap H_t D^p Y = 0$$

to get equality $C^p H_t Y = H_t C^p Y$.

Proposition B.12.

$$C^p H_t Y \cap H_t D^p Y = 0$$

Proof. We imitate the proof of Lemma B.9. Let α be in the intersection. Since $\alpha \in H_t D^p Y$, we may write $\alpha = [\sum_{i>0} d^i x_i]$. Set $y = \sum_{i>0} d^i x_i$, which we assume to be nonzero. Let k be the largest integer with $d^k x_k \neq 0$. Because $\alpha \in C^p H_t Y$, we know $s^k y$ is homologous to zero, so y is homologous to $y - d^k s^k y$. But

$$\begin{aligned} y - d^k s^k y &= \sum_{k \geq i > 0} d^i x_i - \sum_{k \geq i > 0} d^k s^k d^i x_i \\ &= \sum_{k > i > 0} d^i x_i + d^k x_k - d^k s^k d^k x_k - \sum_{k > i > 0} d^k s^k d^i x_i \\ &= \sum_{k > i > 0} d^i (x_i - d^{k-1} s^{k-1} x_k) \end{aligned}$$

so $\alpha = [\sum_{k>i>0} d^i x_i']$. Repeating this tells us that $\alpha = 0$. ☐

Theorem B.13.

$$C^p H_t(Y) = H_t C^p(Y)$$

Proof. Since $C^p H_t(Y) \cap H_t D^p Y = 0$, we have $C^p H_t(Y) \subset H_t C^p Y$. The reverse inclusion was already discussed. ☐

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VITA

Philip Hackney was born in Flemington, New Jersey on March 6, 1982 and moved to west Michigan when he was five. He took math classes at Muskegon Community College while attending Whitehall High School. After graduating high school in 2000, he enrolled at Central Michigan University where he studied computer science and mathematics. In 2004 he received a VIGRE fellowship and began graduate work at Purdue University. There he studied algebraic topology under the direction of Jim McClure. He will earn his doctorate in August 2010 and continue his study of math as a Visiting Assistant Professor in Mathematics at the University of California, Riverside.