Cohomological rigidity of manifolds arisen from right-angled 3-dimensional polytopes

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- Three-dimensional Lens spaces

$$L(p;q_1) \simeq L(p;q_2) \Leftrightarrow q_1 q_2 \equiv \pm n^2 \mod p$$
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- There are infinitely many nonhomeomorphic smooth manifolds with the homotopy type of $\mathbb{C}P^n$ for $n > 2$.

Let γ be the tautological line bundle over $\mathbb{C}P^1$, and let Σ_n be the total space of the projective bundle $P(\underline{\mathbb{C}} \oplus \gamma^{\otimes n})$ for $n \in \mathbb{Z}$. Then, Σ_n is a closed smooth manifold with a smooth effective action of T^2.†

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[Petrie, 1973]

For an oriented smooth manifold M of $\dim_{\mathbb{R}} = 6$, if $H^*(M) \cong H^*(\mathbb{C}P^3)$ and M admits a nontrivial smooth semifree circle action, then M is diffeomorphic to $\mathbb{C}P^3$.

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Definition

A family of closed manifolds is *cohomologically rigid* over $\mathbf{k}$ if manifolds in the family are distinguished up to homeomorphism by their cohomology rings with coefficients in $\mathbf{k}$.

Goal of this talk

We establish cohomological rigidity for particular two families of manifolds of dim 3 and 6 arising from the *Pogorelov class* $\mathcal{P}$ consisting of the polytopes which have right-angled realizations in Lobachevsky space $\mathbb{L}^3$.

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Simple polytopes

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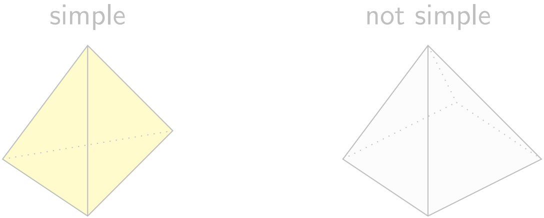
A *polytope* is a convex hull of finite points in $\mathbb{R}^n$.

e.g.) A polygon is a 2-dimensional polytope.
Platonic solids are 3-dimensional polytopes.

Definition

An n-polytope is *simple* if every vertex is the intersection of precisely n facets, codimension-1 faces.



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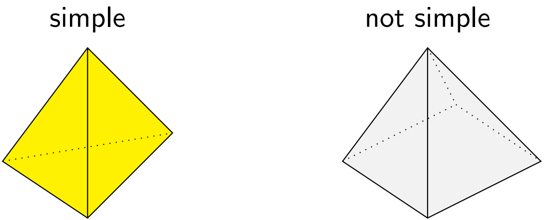
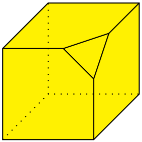
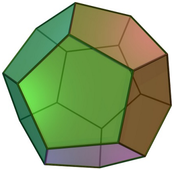


k-belts

For $k \geq 3$, a *k-belt* in a simple 3-polytope is the set of facets k facets, $\mathcal{B}_k$, such that the union of all the facets in $\mathcal{B}_k$ is homotopy equivalent to S^1 and any union of $k-1$ facets in $\mathcal{B}_k$ is contractible.



There are one 3-belt and three 4-belts.



There is neither 3-belt nor 4-belt.

‡

‡The image of dodecahedron is from Wikipedia.

Pogorelov class

Definition

The *Pogorelov class* $\mathcal{P}$ consists of simple 3-polytopes $P \neq \Delta^3$ without 3-and 4-belts.

- This class includes mathematical fullerene, i.e. simple 3-polytopes with only pentagonal or hexagonal facets. (e.g., dodecahedron, truncated icosahedron)
- The number of combinatorially different fullerenes with p_6 hexagonal facets grows as p_6^9. [Thurston, 1998]

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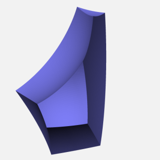
Right-angled 3-dimensional polytopes

[Pogorelov 1967, Andreev 1970]

A combinatorial 3-polytope can be realized as a right-angled polytope in Lobachevsky space $\mathbb{L}^3$ if and only if it is a simple polytope without 3-and 4-belts. Furthermore, such a realization is unique up to isometry.



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§ http://bulatov.org/math/1101/webtalk.html

Hyperbolic 3-manifolds

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[Davis-Januszkiewicz, 1991]

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Let P be a simple n-polytope with facets $F_1, \ldots, F_m$.
Consider a function $\lambda_d\colon \{F_1, \ldots, F_m\} \to \mathcal{R}_d^n$ satisfying

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$$M^{dn}(P, \lambda_d) = P \times G_d^n / \sim,$$

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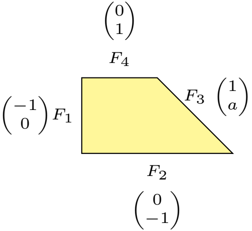
Examples

- A hyperbolic 3-manifold of Löbell type is appeared when $d = 1$ and P is a right-angled 3-polytope.
- Every projective smooth toric variety is appeared when $d = 2$ and P is a Delzant polyope[†]; e.g. Hirzebruch surface Σ_a is



- The manifold $\mathbb{C}P^2 \# \mathbb{C}P^2$ is appeared when $d = 2$, $P = \square$, and $\lambda_2(F_1) = \begin{pmatrix}-1\\0\end{pmatrix}$, $\lambda_2(F_2) = \begin{pmatrix}0\\-1\end{pmatrix}$, $\lambda_2(F_3) = \begin{pmatrix}1\\2\end{pmatrix}$, $\lambda_2(F_4) = \begin{pmatrix}1\\1\end{pmatrix}$

[†] An n-dimensional convex polytope is said to be a *Delzant polytope* if the (outward) normal vectors to the facets meeting at each vertex form an integral basis of $\mathbb{Z}^n$.

Small covers and Quasitoric manifolds

[Davis-Januszkiewicz, 1991]

A closed smooth manifold M of dimension $2n$ (resp. n) is called a *quasitoric manifold* M (resp. *small cover*) if it has a smooth action of T^n (resp. $\mathbb{Z}_2^n$) such that

1. the action of T^n (resp. $\mathbf{Z}_2^n$) is locally standard, and
2. there is a projection $\pi\colon M \to P$ such that the fibers of π are the T^n-orbits (resp. $\mathbf{Z}_2^n$-orbits),

where P is a simple polytope of dimension n.

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The natural action of G_d^n on $\mathbb{F}_d^n$ is called *the standard n-dimensional representation*.

Quasitoric manifolds and small covers

For a quasitoric manifold $M = M^{2n}(P, \lambda_2)$, there is an involution τ on M whose fixed point set is the small cover $M^n(P, \lambda_1)$, where λ_1 is the mod 2 reduction of λ_2.

For the Hirzebruch surface Σ_a, if a is even (resp. odd), then the corresponding small cover is the torus T^2 (resp. the Klein bottle).

Equivalent quasitoric manifolds

Two quasitoric manifolds M and M' over P are *equivalent* if there exist a homeomorphism $f\colon M \to M'$ and an automorphism θ of T^n such that $f(g \cdot x) = \theta(g) \cdot f(x)$ for every $x \in M$ and every $g \in T^n$ and f covers the identity on P.

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Let Λ be the matrix whose ith column is $\lambda(F_i)$. Then $M(P,\lambda)$ and $M(P',\lambda')$ are equivalent if and only if

1. there is a combinatorial equivalence $P \approx P'$ preserving the ordering of facets, and
2. $\Lambda' = A\Lambda B$, where $A \in \mathrm{GL}_n(\mathbb{Z})$ and B is an $m \times m$ diagonal matrix with ± 1 on the diagonal.

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Cohomology of quasitoric manifolds

The cohomology ring of a quasitoric manifold $M = M(P, \lambda)$ is

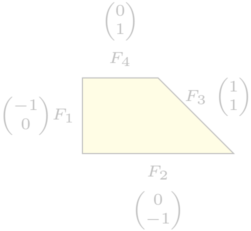
$$H^*(M(P,\lambda)) \cong \mathbb{Z}[v_1, \ldots, v_m]/\mathcal{I}_P + \mathcal{J}_\lambda,\ \deg(v_i) = 2,$$

where

$$\mathcal{I}_P = \langle v_{i_1} \cdots v_{i_k} \mid F_{i_1} \cap \cdots \cap F_{i_k} = \emptyset \text{ in } P \rangle$$

and

$$\mathcal{J}_\lambda = \left\langle \sum_{i=1}^{m} \langle \boldsymbol{\lambda}_i, \mathbf{x} \rangle v_i \,\middle|\, \mathbf{x} \in \mathbb{Z}^n \right\rangle.$$



$$\begin{aligned} &H^*(M(P,\boldsymbol{\lambda})) \\ &\cong \mathbb{Z}[v_1, \ldots, v_4]/\langle v_1v_3, v_2v_4, v_1 - v_3, v_2 - v_3 - v_4 \rangle \\ &\cong \mathbb{Z}[v_3, v_4]/\langle v_3^2, v_4(v_3 + v_4) \rangle \end{aligned}$$

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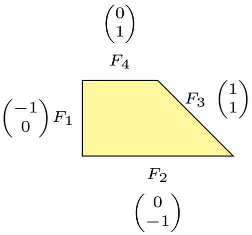
$$\mathcal{J}_\lambda = \left\langle \sum_{i=1}^{m} \langle \boldsymbol{\lambda}_i, \mathbf{x} \rangle v_i \,\middle|\, \mathbf{x} \in \mathbb{Z}^n \right\rangle.$$



$$\begin{aligned} &H^*(M(P, \boldsymbol{\lambda})) \\ &\cong \mathbb{Z}[v_1, \ldots, v_4]/\langle v_1v_3, v_2v_4, v_1 - v_3, v_2 - v_3 - v_4\rangle \\ &\cong \mathbb{Z}[v_3, v_4]/\langle v_3^2, v_4(v_3 + v_4)\rangle \end{aligned}$$

Moment-angle manifold

Let P be a simple n-polytope with facets $F_1, \ldots, F_m$.
Let T_i be the coordinate circle subgroup of T^m corresponding to F_i.
Then for each face $F = \cap_j F_j \neq \emptyset$ of P, we set $T_F = \prod_j T_j$.

Definition

The *moment-angle manifold* corresponding to P is

$$\mathcal{Z}_P = P \times T^m / \sim,$$

where $(x,t) \sim (x',t') \quad \Leftrightarrow \quad x = x' \ \& \ t^{-1}t' \in T_{F(x)}$. Here $F(x)$ is the face containing x in its interior.

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$$\mathcal{Z}_{\Delta^n} = S^{2n+1} \text{ and } \mathcal{Z}_{\prod_{i=1}^k \Delta^{n_i}} = \prod_{i=1}^k S^{2n_i+1}.$$

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Relationship between $M(P, \lambda)$ and $\mathcal{Z}_P$

The matrix $\Lambda = \begin{pmatrix} \boldsymbol{\lambda}_1 & \cdots & \boldsymbol{\lambda}_m \end{pmatrix}$ corresponding to λ induces a surjective homomorphism $\overline{\lambda} : T^m \to T^n$.

$\Longrightarrow \ker(\overline{\lambda})$ is an $(m-n)$-dimensional subtorus of T^m.

Theorem [Davis-Januszkiewicz]

The subtorus $\ker(\overline{\lambda})$ acts freely on $\mathcal{Z}_P$, thereby defining a principal T^{m-n}-bundle $\mathcal{Z}_P \to M(P, \lambda)$.

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The following matrix defines a characteristic function on the standard simplex Δ^n

$$\Lambda = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix}_{n\times(n+1)}.$$

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Cohomology of moment-angle manifolds

Recall that $H^*(M(P,\lambda)) = \mathbb{Z}[v_1, \ldots, v_m]/\mathcal{I}_P + \mathcal{J}_\lambda$.
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Summary

Given a simple polytope P,

- there is a moment-angle manifold $\mathcal{Z}_P$ of dimension $n+m$;
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  - there is a quasitoric manifold $M(P,\lambda_2)$ of dimension $2n$,
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- The quasitoric manifold $M(P,\lambda)$ is $\mathcal{Z}_P/\ker(\overline{\lambda_2})$.

Goal of this talk

We establish cohomological rigidity for particular two families of manifolds of dim 3 and 6 arising from the *Pogorelov class* $\mathcal{P}$ consisting of the polytopes which have right-angled realizations in Lobachevsky space $\mathbb{L}^3$.

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Rigidity problems

In 2006, Masuda and Suh introduced the following problem.

Cohomological rigidity problems for quasitoric manifolds

If two quasitoric manifolds M and M' have the same cohomology ring with integral coefficients, are they homeomorphic? In other words, is the family of quasitoric manifolds cohomologically rigid?

This problem is still OPEN. There is no counter example, but there are many results which support the affirmative answer.

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Known results

1. Quasitoric manifolds of $\dim_{\mathbb{R}} \leq 4$ [Orlik-Raymond (1970)]
2. $\prod_{i=1}^{m} \mathbb{C}P^{n_i}$ [Masuda-Panov (2008), Choi-Masuda-Suh (2010)]
3. Projective smooth toric varieties with second Betti number 2 [Choi-Masuda-Suh (2010)]
4. Quasitoric manifolds with second Betti number 2 [Choi-P-Suh (2012)]
5. Quasitoric manifolds over the cube I^3 and dual cyclic polytopes [Hasui (2015)]
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⋮

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Let $\mathcal{Z}_{P_1}$ and $\mathcal{Z}_{P_2}$ be two moment-angle manifolds whose (bigraded) cohomology rings are isomorphic. Are they homeomorphic? In other words, is the family of moment-angle manifolds cohomologically rigid?

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Cohomological rigidity for small covers

- Two small covers N and N' over the n-cube I^n are diffeomorphic if and only if $H^*(N;\mathbb{Z}_2) \cong H^*(N';\mathbb{Z}_2)$. [Kamishima-Masuda, 2009]
- There are many pair of small covers N and N' over the product of simplices such that $H^*(N;\mathbb{Z}_2) \cong H^*(N';\mathbb{Z}_2)$ but N and N' are not homotopy equivalent. [Masuda, 2010]

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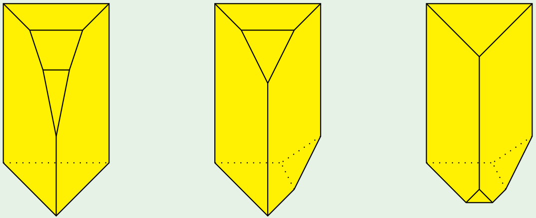
Example

Note that $M(P,\lambda) \cong M(P',\lambda')$ or $\mathcal{Z}_{P_1} \cong \mathcal{Z}_{P_2}$ does not imply that the polytopes P_1 and P_2 are combinatorially equivalent.

The orbit space of $\mathbb{C}P^3 \# 3\overline{\mathbb{C}P^3}$ is a three times vertex-cut of Δ^3.



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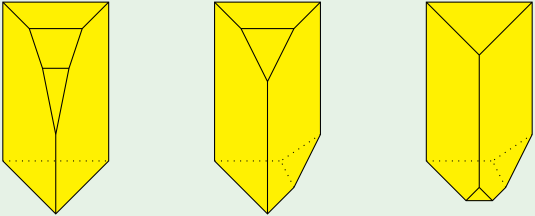

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Rigidity problems for polytopes

Definition [Masuda-Suh]

A simple polytope P is said to be *C-rigid* if it satisfies any of the following

- there is no quasitoric manifold whose orbit space is P; or
- there exists a quasitoric manifold whose orbit space is P, and whenever there exists a quasitoric manifold M' over another polytope P' with a graded ring isomorphism $H^*(M) \cong H^*(M')$, there is combinatorial equivalence $P \approx P'$.

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Over the Pogorelov class

Theorem [Fan-Ma-Wang]

The polytopes in $\mathcal{P}$ are B-rigid.

Note that

- Every simple polytope of dimension 3 admits a characteristic function by the Four Color Theorem. $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \sum_{i=1}^{3} \mathbf{e}_i)$

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Cohomological rigidity

Lemma [Fan-Ma-Wang]

Consider the cohomology classes

$$\mathcal{T}(P) = \{\pm[u_i v_j] \in H^3(\mathcal{Z}_P) \mid F_i \cap F_j = \emptyset\}.$$

If $P \in \mathcal{P}$, then for any cohomology ring isomorphism $\psi\colon H^*(\mathcal{Z}_P) \to H^*(\mathcal{Z}_{P'})$, we have $\psi(\mathcal{T}(P)) = \mathcal{T}(P')$.

Lemma

Consider the set of cohomology classes

$$\mathcal{D}(M) = \{\pm[v_i] \in H^2(M) \mid i = 1, \dots, m\}.$$

If $P \in \mathcal{P}$ and M' is a quasitoric manifold over P', then for any cohomology ring isomorphism $\varphi\colon H^*(M) \to H^*(M')$ we have $\varphi(\mathcal{D}(M)) = \mathcal{D}(M')$. Moreover, the set $\mathcal{D}(M)$ is mapped bijectively under φ.

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Theorem

Let $M = M(P, \lambda)$ and $M' = M(P', \lambda')$. Assume that P belongs to the Pogorelov class $\mathcal{P}$. Then the following are equivalent.

1. $H^*(M)$ and $H^*(M')$ are isomorphic;
2. M and M' are diffeomorphic; and
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Remark

Let Σ_n and Σ_m be Hirzebruch surfaces. Then

- Σ_n and Σ_m are diffeomorphic if and only if $n \equiv m \mod 2$, and
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2. $\pi_1(N)$ and $\pi_1(N')$ are isomorphic;
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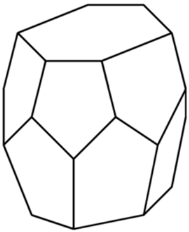
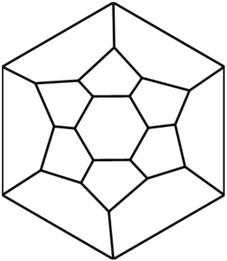
Vesnin's conjecture

For $k \geq 5$, let Q_k be a simple 3-polytope with top and bottom k-gonal facets and $2k$ pentagonal facets forming two k-belts around the top and bottom.

Vesnin's conjecture

The manifolds $N(Q_k, \chi)$ and $N(Q'_k, \chi')$ are isometric if and only if the 4-colorings χ and χ' are equivalent.





Vesnin's conjecture

Vesnin (1987), Magulis (1974), Antonlin-Camarena, Maloney, Gregory, Roland (2009) proved the Vesnin's conjecture except for $k = 6, 8$.

Corollary

The hyperbolic manifold $N(Q_k, \chi)$ and $N(Q_k, \chi')$ defined by regualr 4-colorings of the polytope Q_k, $k \geq 5$, are isometric if and only if the 4-colorings χ and χ' are equivalent.

Cohomological rigidity for 6-dim'l quasitoric manifolds

Theorem [Wall, Jupp]

Let $\varphi\colon H^*(N) \to H^*(N')$ be an isomorphism of the cohomology rings of smooth closed simply connected 6-dimensional manifolds N and N' with $H^3(N) = H^3(N') = 0$. Suppose that

1. $\varphi(w_2(N)) = w_2(N')$, where $w_2(N) \in H^2(N; \mathbb{Z}_2)$ is the second Stiefel-Whiteny class; and
2. $\varphi(p_1(N)) = \varphi(p_1(N'))$, where $p_1(N) \in H^4(N)$ is the first Pontryagin class.

Then the manifolds N and N' are diffeomorphic.

Cohomological rigidity for 6-dim'l quasitoric manifolds

Lemma [Choi-Masuda-Suh]

Suppose that the ring $H^*(N;\mathbb{Z}_2)$ is generated by $H^k(N;\mathbb{Z}_2)$ for some $k > 0$. Then any cohomology ring isomorphism $\varphi\colon H^*(N;\mathbb{Z}_2) \to H^*(N';\mathbb{Z}_2))$ preserves the total Stiefel-Whitney class, i.e., $\varphi(w(N)) = w(N')$.

Corollary

The family of 6-dimensional quasitoric manifolds is cohomologically rigid if any cohomology ring isomorphism between them preserves the first Pontryagin class.

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Thank you very much!