The Small Object Argument

Let $\mathcal{K}$ be a set of maps of a category $\mathcal{M}$

has all colimits.

Goal: factor $f: x \to y$ in $\mathcal{M}$ as

$$\begin{tikzcd} x \arrow[r, "l"] \arrow[rr, bend right, "f"'] & \bullet \arrow[r, "r"] & y \end{tikzcd}$$

with $r \in \mathcal{K}^{\boxslash} =: \mathcal{R}$, $l \in {}^{\boxslash}\mathcal{R} =: \mathcal{L}$

First attempt:

Let S be the set of all commutative squares:

$$\left\{ \begin{tikzcd} a \arrow[r] \arrow[d, "\mathcal{K} \ni"'] & x \arrow[d, "f"] \\ b \arrow[r] & y \end{tikzcd} \right\} \rightsquigarrow \begin{tikzcd} \coprod_S a_i \arrow[r] \arrow[d] & x \arrow[d, "f"] \\ \coprod_S b_i \arrow[r] & y \end{tikzcd}$$

commutative square

Since $\mathcal{K} \subseteq \mathcal{L}$ and $\mathcal{L}$ is saturated, the left arrow is in $\mathcal{L}$.

As $\mathcal{L}$ is saturated, we can take a pushout to get another map $Lf \in \mathcal{L}$.

$$\begin{tikzcd} \coprod_S a_i \arrow[r] \arrow[d] & x \arrow[d, "Lf"] \\ \coprod_S b_i \arrow[r] & \bullet \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \rightsquigarrow \begin{tikzcd} \coprod_S a_i \arrow[r] \arrow[d] & x \arrow[d, "Lf"] \arrow[ddr, bend left, "f"] & \\ \coprod_S b_i \arrow[r] \arrow[drr, bend right] & \bullet \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "Rf"] & \\ & & y \end{tikzcd}$$

From the pushout we obtain a map Rf.

We have produced a factorization

$$\begin{tikzcd} x \arrow[r, "Lf"] \arrow[rr, bend right, "f"'] & \bullet \arrow[r, "Rf"] & y \end{tikzcd}$$

but we do not know if $Rf \in \mathcal{R}$.

Iterate first attempt

$$\begin{tikzcd} x = x_0 \arrow[r, "Lf"] \arrow[drrr, "f"'] & x_1 \arrow[r, "LRf"] \arrow[drr, "Rf"'] & x_2 \arrow[r, "LR^2f"] \arrow[dr, "R^2f"] & x_3 \arrow[r] \arrow[d, "R^3f"] & \cdots\cdots & x_\omega = \operatorname{colim}_n x_n \arrow[dll, "R^\omega f"] \\ & & & y & & \end{tikzcd}$$

- we regard $R^n f = R R^{n-1} f$ as better and better approximations to an element in $\mathcal{R}$.
- as $\mathcal{L}$ is saturated, the transfinite composition $x_0 \longrightarrow x_\omega$ of maps in $\mathcal{L}$ is again in $\mathcal{L}$.

$$\boxed{\text{Is } R^\omega f \in \mathcal{R} = \mathcal{K}^{\boxslash} \text{ ?}}$$

Given a square

$$\begin{tikzcd} a \arrow[r] \arrow[d, "\mathcal{K} \ni"'] \arrow[dr, phantom, "\heartsuit"] & x_\omega \arrow[d, "R^\omega f"] \\ b \arrow[r] & y \end{tikzcd}$$

can we construct a lift $b \longrightarrow x_\omega$?

Hypothesis:
a is compact whenever $a \longrightarrow b$ is in $\mathcal{K}$.

$$\mathcal{M}(a, \operatorname{colim}_n z_n) \cong \operatorname{colim}_n \mathcal{M}(a, z_n)$$

for every sequence $\omega \xrightarrow{z} \mathcal{M}$

Guarantees we can factor the top map through some finite stage n

$$\begin{tikzcd} a \arrow[r, "u"] \arrow[d] & x_n \arrow[r] \arrow[d, "R^n f"] & x_\omega \arrow[d, "R^\omega f"] \\ b \arrow[r, "v"] & y \arrow[r, "="] & y \end{tikzcd}$$

The first attempt construction applied to $R^n f$ is:

$$\begin{tikzcd} a \arrow[r] \arrow[d] & \coprod_{S_n} a_i \arrow[r] \arrow[d] & x_n \arrow[d, "LR^n f"] \arrow[ddr, bend left, "R^n f"] & \\ b \arrow[r] & \coprod_{S_n} b_i \arrow[r] \arrow[drr, bend right] & x_{n+1} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, "R^{n+1}f"] & \\ & & & y \end{tikzcd}$$

Using that

$$\left(\begin{tikzcd} a \arrow[r, "u"] \arrow[d] & x_n \arrow[d, "R^n f"] \\ b \arrow[r, "v"] & y \end{tikzcd} \right) \in S_n$$

$$\begin{tikzcd} a \arrow[r] \arrow[d] \arrow[rr, bend left, "u"] & \coprod_{S_n} a_i \arrow[r] \arrow[d] & x_n \arrow[d, "LR^n f"] \arrow[ddr, bend left, "R^n f"] & \\ b \arrow[r] \arrow[drrr, bend right, "v"'] & \coprod_{S_n} b_i \arrow[r] \arrow[drr, bend right] & x_{n+1} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, "R^{n+1}f"] & \\ & & & y \end{tikzcd}$$

$$\begin{tikzcd} a \arrow[r, "u"] \arrow[d] & x_n \arrow[r, "LR^n f"] \arrow[d, "R^n f"] & x_{n+1} \arrow[r] \arrow[d, "R^{n+1} f"] & x_\omega \arrow[d, "R^\omega f"] \\ b \arrow[r, "v"] \arrow[urr] & y \arrow[r, "="] & y \arrow[r, "="] & y \end{tikzcd}$$

The composite

$$\begin{tikzcd} & x_{n+1} \arrow[r] & x_\omega \\ b \arrow[ur] & & \end{tikzcd}$$

is a lift in $\heartsuit$

So $R^\omega f \in \mathcal{R}$, and we have the desired factorization

$$\begin{tikzcd} x_0 \arrow[r] \arrow[dr, "f"'] & x_\omega \arrow[d, "R^\omega f"] \\ & y \end{tikzcd}$$

♡PJH