SPHERES AS d-SEGAL SETS

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Abstract. We compute for which d the simplicial sets $\Delta^n/\partial\Delta^n$ and $\partial\Delta^{n+1}$ are d-Segal sets.

In this note we use the same conventions and notations for simplicial sets and higher Segal conditions as in [2]. In particular, we use the Wiggle Lemma implicitly when proving simplicial sets are d-Segal.

Origin. In an appendix of [3], we showed that $\Delta^n/\partial\Delta^n$ is lower $(4n-1)$-Segal (as a side-effect of a computation for symmetric versions of the spheres), but suspected that one could do much better. I proved the $n = 2$ case of Theorem 5 in September 2025 while visiting Melbourne. On July 19, 2026, on a whim, I described the problem to Claude Fable, which wrote code to search small cases; these computations eventually led to the counterexamples of Example 2 and Example 3. On July 22, I asked Fable for a proof of Theorem 4 and Theorem 5; at a cost of \$7.13 it produced an incomplete argument[1] that nonetheless contained a genuine strategy. With a Claude Opus model helping me recast this strategy in the language of [2], I turned this into proofs of those results. Subsequently, I asked Opus to carry out computations for $\partial\Delta^{n+1}$, and it suggested the statements and proof strategies in Section 2.

1. Simplex modulo boundary

In this section we consider, for $n \geq 1$, the simplicial model for the sphere $\Delta^n/\partial\Delta^n$. For which d is this upper and/or lower d-Segal? When $n = 1$, this circle is the nerve of the partial monoid $\{1, x\}$ (with xx undefined), hence is 2-Segal [1]. It is evidently not lower 1-Segal, and is not constant so is not upper 1-Segal. In general:

Theorem 1. *Let $n > 1$ be an integer. Then $\Delta^n/\partial\Delta^n$ is $2n$-Segal, but not lower $(2n-1)$-Segal. It is upper $(2n-1)$-Segal if and only if n is odd.*

This follows from Example 2, Example 3, Theorem 4, and Theorem 5.

When talking about simplices in $X = \Delta^n/\partial\Delta^n$ (at $S \in \Delta$), we just regard them as functions $S \to [n]$, though strictly speaking we're identifying all of the nonsurjective, or *trivial*, functions $S \to [n]$ to a single basepoint $* \in X(S)$. We'll write $f \simeq g \colon S \to [n]$ to indicate that either $f = g$ or that f and g are both nonsurjective. We'll also call surjective functions *nontrivial*. Notice that if $e_i f_j = e_j f_i$ is surjective, then there exists a unique F such that $e_i F = f_i$ and $e_j F = f_j$ since Δ^n is lower 1-Segal.

Example 2. Let $F, G \colon [2n] \to [n]$ be given by the formulas $F(t) = \max(0, t - n)$ and $G(t) = \min(t, n)$. The function F restricts to a bijection $[n+1, 2n] \cong [1, n]$

Date: July 26, 2026 (Version 1).

[1] In its words, "the two lemmas above are the honest residue between 'convincing argument + exhaustive verification' and 'theorem.'"

(with other elements sent to 0), and G restricts to the identity on $[0, n-1]$ (with other elements sent to n). When n is even let I be the set of odd elements in $[2n]$, and when n is odd let I be the set of even elements in $[2n]$. In either case define $f_i \coloneqq e_i F$ for $i < n$ and $f_i = e_i G$ for $i > n$ (noting that $n \notin I$). Each f_i is surjective, and this is a compatible collection of nontrivial elements in X. Compatibility is clear for $I \cap [0, n]$ and for $I \cap [n, 2n]$, and cross terms $e_i f_j$ are trivial when i and j are on opposite sides of n (since $e_j F = *$ when $j > n$ and $e_i G = *$ when $i < n$). Now if $H \in X_{2n}$ is any element with $e_i H = f_i$ for all i, then for each $i < n$ we have $H(n) = f_i(n) = F(n) = 0$, while for each $i > n$ we have $H(n) = f_i(n) = G(n) = n$, an impossibility. This establishes that $\Delta^n / \partial\Delta^n$ is not upper $(2n-1)$-Segal when n is even, and $\Delta^n/\partial\Delta^n$ is not lower $(2n-1)$-Segal when n is odd.

Example 3. Suppose n is even, and let $F, G \colon [2n] \to [n]$ be given by the formulas

$$F(t) = \min(\max(0, t-n+1), n)$$
$$G(t) = \max(t-n, \min(t, n-1)).$$

Expanding, these functions act as follows:

t	0	1	$\cdots$	$n-1$	n	$n+1$	$\cdots$	$2n-2$	$2n-1$	$2n$
$F(t)$	0	0	$\cdots$	0	1	2	$\cdots$	$n-1$	n	n
$G(t)$	0	1	$\cdots$	$n-1$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	n

Let $I \subset [2n]$ be the set of even elements, $I_F = I \cap [0, n-2] \cup \{2n\}$, and $I_G = I \cap [n, 2n-2]$. Set $f_i = e_i F$ for $i \in I_F$ and $f_i = e_i G$ for $i \in I_G$. This is a compatible collection of elements – we only need to check the cross terms, and these vanish since $e_i G = *$ for $i \in I_F$ and $e_i F = *$ for $i \in I_G$. If $H \in X_{2n}$ is an element with $e_i H = f_i$ for all $i \in I$, then for $i \in I_F$ we have $H(1) = f_i(1) = F(1) = 0$ and for $i \in I_G$ we have $H(1) = f_i(1) = G(1) = 1$, an impossibility. We conclude that $\Delta^n/\partial\Delta^n$ is not lower $(2n-1)$-Segal when n is even.

If $I \subset S$ is gapped with more than $\frac{n+2}{2}$ elements and $\{f_i\}_{i \in I}$ consists of trivial elements, then $* \in X(S)$ is the unique filler. Indeed, suppose $F \colon S \to [n]$ is surjective. If $\{i\} = F^{-1}F(i)$ for all $i \in I$, we also need somewhere outside of $\{F(i) \mid i \in I\} \subseteq [n]$ to send elements *between* the elements of I, implying $|I| + (|I| - 1) \leq n + 1$.

Theorem 4. *If $n > 1$ is odd, then $X = \Delta^n/\partial\Delta^n$ is upper $(2n-1)$-Segal.*

Proof. Let I be the set of odd elements in $[2n]$ and for a fixed $p \geq 2n$ declare $S = [p]$. Let $\{f_i\}_{i \in I}$ be a compatible collection of $(p-1)$-simplices in $X = \Delta^n/\partial\Delta^n$ (i.e. $f_i \in X(S_i)$ with $e_i f_j \simeq e_j f_i$ for $i \neq j$ in I).

Define the following relation on I: we say $i \sim j$ if $e_i f_j$ is nontrivial, which implies $e_i f_j = e_j f_i$. As Δ^n is a category (and hence lower 1-Segal), $i \sim j$ implies there exists a unique F such that $e_i F = f_i$ and $e_j F = f_j$. Let's check that $\sim$ is transitive, and to that end suppose also $j \sim k$ and let G be the element with $e_j G = f_j$ and $e_k G = f_k$. By compatibility we either have $e_i f_k = e_k f_i$ or we have $e_i f_k$ is trivial. If $e_i f_k$ is trivial, then $f_k(i)$ is only hit once by f_k. This means $f_k(i \pm 1) = f_k(i) \pm 1$ by surjectivity, and the fact that $i \pm 1 \notin I$, so in particular is different from k. But $f_k(i \pm 1) = G(i \pm 1) = f_j(i \pm 1) = F(i \pm 1) = f_i(i \pm 1)$. This implies that $f_k(i)$ is not in the image of f_i, contrary to our assumption that f_i is nontrivial. Now each equivalence class E of I having more than one element will have a unique F such that $e_i F = f_i$ for all $i \in E$ (since Δ^n is lower 1-Segal, and hence lower $(|E|-1)$-Segal).

Suppose $i \in I$ is such that f_i is nontrivial. We'd like to know how large its equivalence class E is. Define the following sets:

$$J = \{j \in I \mid e_j f_i = *\}$$
$$K = \{k \in I \mid e_k f_i \text{ surjective}\},$$

which satisfy $I = \{i\} \amalg J \amalg K$ and $E = K \amalg \{i\}$.

Notice for $j \in J$ that $f_i(j-1) < f_i(j) < f_i(j+1)$, and hence $f_i(j+1) - f_i(j-1) = 2$. Now

$$2|J| = \sum_{j \in J} f_i(j+1) - f_i(j-1) \leq \sum_{t \in I} f_i(t+1) - f_i(t-1) = f_i(2n) - f_i(0) \leq n.$$

Thus $|J| \leq \frac{n}{2}$. Since n is odd and $E \amalg J = I$ has n elements, E has at least $(n+1)/2$ elements. Since E contains more than half of the elements of I, there cannot be any other equivalence class containing an i with f_i nontrivial. By the preceding paragraph we know there is a unique F such that $e_i F = f_i$ for all $i \in E$.

It just remains to show that $e_j F$ is trivial for all $j \in J$ (since we know that f_j is trivial), but this is about $e_j F(j \pm 1) = f_i(j \pm 1) = (e_j f_i)(j \pm 1)$ being two apart. Thus we have established uo_p^{n-1}. □

In particular, when n is odd then $\Delta^n / \partial \Delta^n$ is $2n$-Segal.

Theorem 5. *If $n > 1$ is even, then $X = \Delta^n / \partial\Delta^n$ is $2n$-Segal.*

Proof. Since $X \cong X^{\mathrm{op}}$, it suffices to show that X is lower $2n$-Segal. Let $I \subset [2n+1]$ be the set of odd elements (an $n+1$ element set), considered as a gapped subset of $S = [p]$ for a fixed $p \geq 2n+1$. Let $\{f_i \in X(S_i)\}_{i \in I}$ be a compatible collection of $(p-1)$-simplices.

The argument proceeds as in the proof of Theorem 4. We again define a relation $\sim$ on I, which is again transitive. The proof of this works even when $p = 2n+1$ if we assume $i < k$, which is fine since $e_i f_k \simeq e_k f_i$.

We define J and K in an identical way, and we wish to show that $|E| \geq \frac{n}{2} + 1$, i.e. that E contains more than half of the elements of the $(n+1)$ element set I. This implies that there can be at most one nontrivial equivalence class. If $p > 2n+1$ we obtain the inequality

$$2|J| = \sum_{j \in J} f_i(j+1) - f_i(j-1) \leq \sum_{t \in I} f_i(t+1) - f_i(t-1) = f_i(2n+2) - f_i(0) \leq n$$

exactly as before, so $|J| \leq \frac{n}{2}$ and hence $|E| \geq \frac{n}{2} + 1$. If $p = 2n+1$ is not a member of J, then we have (instead summing over I_p the set of odds less than p)

$$2|J| = \sum_{j \in J} f_i(j+1) - f_i(j-1) \leq \sum_{t \in I_p} f_i(t+1) - f_i(t-1) = f_i(2n) - f_i(0) \leq n,$$

and may again conclude $|E| \geq \frac{n}{2} + 1$. Finally, if $p = 2n+1$ is a member of J, then we compute

$$2|J_p| = \sum_{j \in J_p} f_i(j+1) - f_i(j-1) \leq \sum_{t \in I_p} f_i(t+1) - f_i(t-1) = f_i(2n) - f_i(0) = n-1,$$

implying $|J_p| \leq \frac{n-1}{2}$, and, since n is even, $|J| \leq \frac{n}{2}$. Thus we again have $|E| \geq \frac{n}{2} + 1$.

Since $|E| - 1 \geq \frac{n}{2} \geq 1$ and Δ^n is lower 1-Segal, there exists a unique F such that $e_i F = f_i$ for all $i \in E$. As in the proof of Theorem 4 we see that $e_j F$ is trivial for all $j \in J$, unless $p = 2n+1$ and $j = p$. In this last case, since we know $e_p f_i$ is trivial, we have $f_i(2n) = n-1$, but $F(2n) = f_i(2n)$, hence e_pF does not have n in its image, hence is trivial. We conclude that X is lower $2n$-Segal. □

2. Simplex boundaries

Proposition 6. *If $n \geq 0$, then $\partial\Delta^{n+1}$ is $(n{+}1)$-coskeletal but not n-coskeletal.*

Proof. It's clear that it's not n-coskeletal, as the identity $\partial\Delta^{n+1} \to \partial\Delta^{n+1}$ does not have a filler $\Delta^{n+1} \to \partial\Delta^{n+1}$. If $n = 0$ then $\partial\Delta^{n+1} = \partial\Delta^1$ is discrete, hence 1-coskeletal. We now assume $n \geq 1$. Suppose $m > n+1$, $S = [m]$, and $f_i \in X(S_i)$ is a compatible collection (ranging over all $i \in S$). There is a unique $F\colon [m] \to [n+1]$ with $e_iF = f_i$ for all i, this is using that Δ^{n+1} is 2-coskeletal since it's a category, and $m > n+1 \geq 2$. If F is surjective, then there exists a k such that $F^{-1}(k)$ has more than one element; if $i \in F^{-1}(k)$, then $e_iF = f_i$ is surjective, contrary to assumption. Thus $F \in (\partial\Delta^{n+1})_m$. □

Proposition 7. *For $n \geq 1$, $\partial\Delta^{n+1}$ is $(n{+}1)$-Segal but not upper or lower n-Segal.*

Proof. Suppose $X = \partial\Delta^{n+1}$ is not $(n{+}1)$-critical. Then for $S = [n+2]$ or $[n+3]$ we can find a compatible collection whose unique filler in Δ^{n+1} is surjective. We may assume that $I \subset [n+2]$ is either the set of even elements[2] or the set of odd elements.[3] Consider I as a gapped subset of S, and let $\{f_i\}_{i\in I}$ be a compatible collection whose unique filler in $\Delta^{n+1}(S)$ is a surjective function $F\colon S \to [n+1]$. If $S = [n+2]$ then there is exactly one k such that $F^{-1}(k)$ contains two elements; by parity, one must be in I, which then gives $e_iF = f_i$ surjective, contrary to assumption. For $S = [n+3]$, there are two possibilities for the surjective function $F\colon [n+3] \to [n+1]$. If there is a $k < n+1$ having $F^{-1}(k)$ having at least two elements, then $I \cap F^{-1}(k)$ will be nonempty, so we can argue by parity as before. If $F^{-1}(n+1)$ has size three, then it will contain the largest element of I, and we again do the same thing. We've thus established that $\partial\Delta^{n+1}$ is $(n{+}1)$-critical – since it's $(n{+}1)$-coskeletal, it is $(n{+}1)$-Segal by the main theorem of [2].

To see that $\partial\Delta^{n+1}$ is not upper or lower n-Segal, let $F\colon [n+1] \to [n+1]$ be the identity and $I \subset [n+1]$ to be either the set of even elements or the set of odd elements. Then $\{e_iF\}_{i\in I}$ is an appropriate compatible collection of n-simplices in $\partial\Delta^{n+1}$, but its only filler is surjective. (If $n = 1$, the argument against upper 1-Segal is simply that $\partial\Delta^2$ is not constant.) □

References

1. Julia E. Bergner, Angélica M. Osorno, Viktoriya Ozornova, Martina Rovelli, and Claudia I. Scheimbauer, *2-Segal sets and the Waldhausen construction*, Topology Appl. **235** (2018), 445–484. MR 3760213
2. Philip Hackney, *Coskeletality and the higher Segal conditions*, in preparation.
3. Philip Hackney and Justin Lynd, *Higher Segal spaces and partial groups*, arXiv:2507.05437v2 [math.GR].

[2] If n is even this set has size $(n/2+1)+1$ and may contain both endpoints, while if n is odd this set has size $(n+1)/2+1$ and misses $n+2$.

[3] If n is even this set has size $n/2+1$ and misses $0, n+2$, while if n is odd this set has size $(n+1)/2+1$ elements and misses 0 but not $n+2$.