Higher Segal spaces and partial groups

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(joint work with Justin Lynd)

Our purpose is to explain and explore a new connection between the d-Segal spaces of Dyckerhoff and Kapranov and the partial groups of Chermak. The former objects have applications (when $d = 2$) in representation theory, geometry, combinatorics, and elsewhere, and are closely connected to ∞-operads, Span-enriched A_∞-algebras, and operadic categories. The latter objects played a key role in Chermak's proof of the existence and uniqueness of centric linking systems for saturated fusion systems, a major recent result in p-local finite group theory.

Partial groups

Partial groups [C] are akin to groups, but where the n-fold multiplications $G^{\times n} \to G$ are replaced by partial functions. These may be concisely described as 'reduced spiny symmetric sets' by [HL], as we now explain. Let Υ be the category with the same objects $[n] = \{0, 1, \dots, n\}$ as the simplicial category Δ, but with arbitrary functions as morphisms. A *symmetric (simplicial) set* is a functor $X \colon \Upsilon^{\mathrm{op}} \to \mathsf{Set}$. Groupoids may be identified with those symmetric sets X such that the Segal maps

$$X_n \to X_1 \times_{X_0} \underset{n}{\cdots} \times_{X_0} X_1$$

are bijections for all $n \geq 2$. A *spiny symmetric set* is a symmetric set X such that the Segal map is an injection for all $n \geq 2$, and a *partial group* is the same thing as a spiny symmetric set with X_0 a point. The partially-defined n-fold multiplication is defined by the span $X_1^{\times n} \hookleftarrow X_n \to X_1$ where the map on the right is given by the endpoint-preserving map $[1] \to [n]$. Every group G can be considered as a partial group, by identifying it with the associated symmetric set BG.

Every nonempty symmetric subset of BG is a partial group, and many important partial groups arise in this way. (Though not every partial group may be embedded into a group.) For example, $B_{\mathrm{com}}G \subseteq BG$ has n-simplices those $[g_1|\cdots|g_n] \in BG_n = G^{\times n}$ where $g_i g_j = g_j g_i$ for all i, j (see [AG]). Let us give another fundamental class of examples:

Example. Suppose G acts on a set V, and U is a subset of V. Then G 'acts partially' on the set U, and we let E be the simplicial set with n-simplices of the form

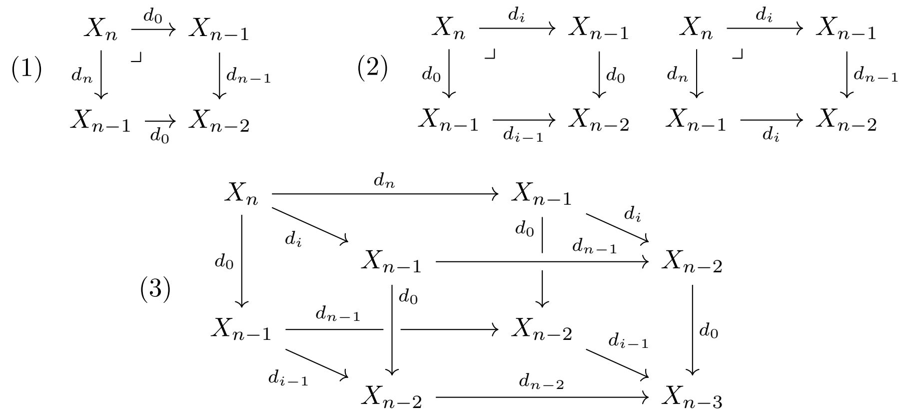
$$u_0 \xrightarrow{g_1} u_1 \xrightarrow{g_2} \cdots \xrightarrow{g_n} u_n$$

with $u_i \in U$ and $g_i \cdot u_{i-1} = u_i$. This E is a groupoid with object set $E_0 = U$, and we let $L \subseteq BG$ be the image of the map $E \to BG$.

For instance, consider the action of G by conjugation on the set V of subgroups of G, and let $U \subseteq V$ be the set of nontrivial subgroups in a fixed Sylow p-subgroup of G. The most important class of partial groups are the *localities*, which are modeled on this situation.

Higher Segal conditions are certain exactness conditions associated to a simplicial object, generalizing the usual Segal condition which underlies some models for $(\infty, 1)$-categories. The 2-Segal conditions first appeared in [DK], while the d-Segal conditions for $d > 2$ are explored in [P, W].

The d-Segal conditions[1] can be phrased in terms of a simplicial object X having a small number of associated cubes being (homotopy) limit cubes (of dimension $\lceil \frac{d}{2} \rceil + 1$). Write $i \ll j$ to mean $i < j - 1$. The 1-Segal condition is that (1) is a pullback for all $n \geq 2$, the 2-Segal condition is that the squares (2) are pullbacks whenever $0 \ll i \ll n$, and the 3-Segal condition is that the cube (3) is cartesian whenever $0 \ll i \ll n$.



For the 4-Segal condition, one replaces the cubes (3) associated with integers $0 \ll i \ll n$ by cubes associated with $0 \ll i \ll j (< n)$ and $(0 <) i \ll j \ll n$. The 5-Segal condition concerns the four dimensional cubes associated to $0 \ll i \ll j \ll n$, and so on. A d-Segal object is automatically $(d+1)$-Segal, so one could wonder about the minimal d (if any) for a simplicial object to be d-Segal. For partial groups, this will turn out to always be odd. Let us give an indication of why this is true.

Theorem (H–Lynd)**.** *If a symmetric set is 2-Segal, then it is 1-Segal.*

Proof. The symmetric group action implies that for $n \geq 3$, square (1) is isomorphic to any of the squares in (2). The $n = 2$ instance of square (1) is a retract of the $n = 3$ instance of square (1). Thus (1) is a pullback for all $n \geq 2$. □

Definition. The degree of a partial group X, denoted $\deg(X)$, is the least positive integer k such that X is $(2k-1)$-Segal.

Groups are precisely the degree 1 partial groups. One can show that $B_{\mathrm{com}}G$ is 3-Segal, hence has degree 1 or 2. There are rich families of partial groups (arising from the example above) attaining arbitrarily high degree.

A primary method for calculating $\deg(X)$ is to consider sufficiently nice actions of X on various sets U. Such an action can be encoded as a map $\rho \colon E \to X$

[1] For d odd, we only consider the *lower* d-Segal conditions.

satisfying certain properties, where E is a groupoid with $E_0 = U$. (This includes $E \to L$ from our example.) This gives rise to a closure operator $A \mapsto \bar{A}$ on E_0 defined in terms of simplices of X which act on all elements of $A \subseteq E_0$. A collection Γ of nonempty subsets of E_0 is *independent* if the set

$$\bigcap_{\substack{\Lambda \subset \Gamma \\ |\Gamma \setminus \Lambda| \leq 1}} \overline{\bigcup \Lambda}$$

is empty, and $h(\rho)$ is defined to be the size of the largest independent $\Gamma \subseteq 2^{E_0}$.

Theorem (H–Lynd)**.** $\deg(X) \leq h(\rho)$.

Corollary. *The degree of a finite partial group is finite.*

Proof. A partial group X is said to be finite just when X_1 is a finite set. Every finite partial group is finite-dimensional as a symmetric set by [HM], and hence has finitely many nondegenerate simplices. The canonical map $E = \coprod_{\mathrm{nd}(X)} \Upsilon^n \xrightarrow{\rho} X$ is a nice action of X on the finite set E_0. It follows that $h(\rho)$ is finite. □

We have now explained the rudiments of the connection between partial groups and higher Segal structures, by realizing partial groups as symmetric simplicial sets. We introduced a new invariant for partial groups – the degree – and a method for producing upper bounds for this invariant. In future work, we will calculate the degree for a number of important classes of examples, providing a source of interesting d-Segal spaces for large d.

References

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