On pasting in $(\infty, 2)$-categories

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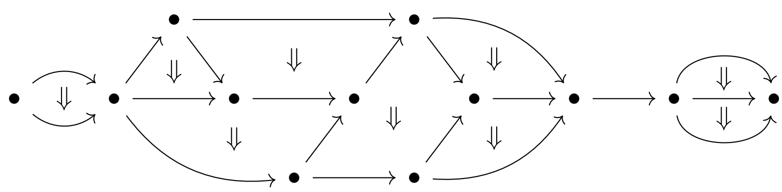
Given a category C and a sequence of morphisms $f_1, f_2, \dots, f_m$ with the target object of f_i equal to the source object of f_{i+1}, there is a unique composite morphism $f_m \circ f_{m-1} \circ \cdots \circ f_1$. If instead C is an $(\infty, 1)$-category, there need not be a unique composite in the usual sense. Instead, there is a *space* parameterizing both the possible compositions $f_m \circ f_{m-1} \circ \cdots \circ f_1$ of the given morphisms, and the relationships among all of these possibilities. Composition in an $(\infty, 1)$-category is *homotopically* unique in the sense that this space is always contractible.

In a 2-category, there are more complicated sorts of compositions possible among collections of 1- and 2-morphisms, which should be familiar to anyone who has worked with natural transformations of functors between categories. For instance, natural transformations between functors can be whiskered along other functors, or composed vertically or horizontally.

$$\begin{tikzcd} C \arrow[r] & D \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & E \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r] \arrow[r, bend left=60] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\Downarrow"] \arrow[r, phantom, shift right=3, "\Downarrow"] & D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & D \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & E \end{tikzcd}$$

One can of course put these basic pictures together into more complicated ones, but the same picture can arise in multiple ways. Power proved in [4] that general pasting in a 2-category is well-defined. *Pasting schemes* are planar directed graphs satisfying certain conditions, such as the following.



A labeling of a pasting scheme by a 2-category X is given by assigning an object to each vertex, a 1-morphism to each directed edge (so that the source and target of the 1-morphism match the labelings of the end vertices), and a 2-morphism to each interior face (whose source and target match the composites of the 1-morphism labelings on the boundaries). Power showed that given such a labeling of a pasting scheme by a 2-category, there is a unique composite 2-morphism in the 2-category.

When working on some problems in $(\infty, 2)$-category theory, we found ourselves reasoning via pasting diagrams, but were surprised to discover that this fundamental result had not yet been established in this context. We sought to rectify this, and proved in [3] that a labeling of a pasting scheme by an $(\infty, 2)$-category X has a homotopically unique composite.

Theorem. *Given a pasting scheme and a labeling of the pasting scheme by an $(\infty, 2)$-category X, the associated space of composites is contractible.*

The labeling in this context is defined in terms of homotopy colimits of the constituent cells, which handles, for instance, the possible ambiguity about 1-morphism composition in the underlying $(\infty, 1)$-category of X. The idea of the proof is to show that the free $(\infty, 2)$-category on a pasting scheme and the free 2-category on the pasting scheme are equivalent as $(\infty, 2)$-categories, and then apply Power's theorem. Establishing this equivalence is not formal, as the functor from 2-categories to $(\infty, 2)$-categories is not cocontinuous. Indeed, our proof of the equivalence relies heavily on delicate calculations in a particular model for $(\infty, 2)$-categories. We work in the simplicial categories model of Lurie, which has the benefit that many of the horizontal compositions come 'for free' and do not need to be added by hand. The actual computations utilize a new sharpening of a result of Thomason about how the nerve functor interacts with pushouts of Dwyer maps between small categories [5, 4.3]. There is a different, earlier proof of this same equivalence (in the same model) in the unpublished PhD thesis [1] of Tobias Columbus, who contacted us after our initial posting; a revised version of this thesis is now readily available [2].

One of course hopes for a similar theorem for pasting in (∞, n)-categories when $n > 2$, though this is a more difficult problem. Indeed, the combinatorial aspects of n-categorical pasting are more subtle, and there are not simple extensions of our methods to higher dimensions.

References

[1] T. Columbus, 2-*Categorical Aspects of Quasi-Categories*, PhD thesis, Karlsruher Institut für Technologie (2017).

[2] T. Columbus, *Pasting in simplicial categories*, preprint arXiv:2106.15861 [math.CT].

[3] P. Hackney, V. Ozornova, E. Riehl, M. Rovelli, *An* $(\infty, 2)$-*categorical pasting theorem*, preprint arXiv:2106.03660 [math.AT].

[4] A. J. Power, *A* 2-*categorical pasting theorem*, J. Algebra, **129**(2) (1990) 439–445.

[5] R. W. Thomason, Cat as a closed model category, Cahiers Topologie Géom. Différentielle, **21**(3) (1980), 305–324.